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Constructive approaches to the rigidity of frameworks

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Chapter 1

Introduction

Contents

La théorie de la rigidité se préoccupe originellement de la rigidité/flexibilité des charpentes. Une charpente est un modèle mathématique décrivant une structure réelle. Une des premières sources de structures qui ont suscité l'étude de la rigidité vient de l'ingénierie des structures. Considérons une structure planaire constituée de barres solides jointes aux extrémités par des jonctions permettant à deux barres jointes de bouger librement dans le plan autour du joint. Une question naturelle est de savoir si la structure peut être déformée. Par exemple, la structure planaire rectangulaire de la Figure 1.1 peut être déformée continûment en une structure planaire non-rectangulaire comme représenté par la figure en pointillés. Nous disons que la structure est flexible. Au contraire, la structure planaire de la Figure 1.2 ne peut pas être déformée continûment en une autre forme sans être cassée. Nous disons que la structure est rigide, ou plus précisément, localement rigide.

Ce type de structures barres-et-joints planaires peut être modélisée par une paire $(G, \mathbf{p})$ formée d'un graphe $G = (V, E)$ et d'une application $\mathbf{p} : V \to \mathbb{R}^2$. Les sommets V de G correspondent aux joints et les arêtes E de G correspondent aux barres.

L'application $\mathbf{p}$ détermine la position des joints. De la même manière, une structure barres-et-joints en dimension 3 (et plus généralement en dimension d) peut être modélisée par une application $\mathbf{p} : V \to \mathbb{R}^3$ (et $\mathbf{p} : V \to \mathbb{R}^d$, respectivement). Nous appelons cette paire une charpente barres-et-joints. Un déplacement continu d'une charpente $(G, \mathbf{p})$ dans $\mathbb{R}^d$ est un déplacement continu des sommets de G dans $\mathbb{R}^d$ qui conserve les longueurs des arêtes. Une charpente est dite localement rigide si tous les déplacements continus de $(G, \mathbf{p})$ préservent aussi la distance entre chaque paire de sommets de G. De manière équivalente, tout déplacement continu de $(G, \mathbf{p})$ est la restriction d'un déplacement congruent de l'espace tout entier.

Déterminer si une charpente est localement rigide est difficile en général. Par contre, on peut s'intéresser aux déplacements instantanés des joints du problème linéarisé. Un déplacement infinitésimal d'un vecteur $\mu(v)$ à chaque sommet $v \in V$ ($\mu(v)$ peut être vu comme la vitesse instantanée du joint) tel que pour chaque arête (i.e. barre) les déplacements instantanés le long de cette arête induits par ce déplacement infinitésimal aux deux extrémités sont identiques, ce qui signifie

$$(p(u) - p(v))(\mu(u) - \mu(v)) = 0 \quad \text{pour tout } uv \in E. \tag{1.0.1}$$

(Voir Figure 3.2.) Une charpente est dite infinitésimalement rigide si tous ses déplacements infinitésimaux peuvent être obtenus à partir des vecteurs de vitesse instantanée de la restriction d'un déplacement congruent de tout l'espace sur V.

En fait, la rigidité infinitésimale est une propriété plus forte que la rigidité locale, mais c'est une bonne alternative à la rigidité locale car il s'avère qu'elles coïncident dans la plupart des cas, à savoir pour les charpentes en position générique. La rigidité infinitésimale offre plus de prise que la rigidité locale puisque elle peut être déterminée par le calcul du rang de la matrice dérivée du système linéaire (1.1.1) pour μ.

L'étude de la rigidité a trouvé des applications pour prédire la flexibilité de protéines.

D'autres notions de rigidité méritent notre attention. La première se préoccupe de l'unicité globale des charpentes, à savoir, si l'ensemble des contraintes de distance imposées aux barres suffit à déterminer la charpente modulo une congruence. L'étude de la rigidité globale a des applications dans la localisation dans des réseaux de capteurs.

Bien que la rigidité globale apparaît très différente de la rigidité locale, elles sont en fait étroitement liées. Il est facile de voir que la rigidité globale implique la rigidité locale. Dans l'autre direction, il est démontré qu'une charpente planaire générique est globalement rigide si et seulement si il faut enlever au moins 3 sommets pour la déconnecter et elle reste localement rigide après la suppression d'une arête quelconque.

En pratique, pour calculer la position des capteurs, on peut utiliser des algorithmes basés sur la programmation semi-définie.

Cependant, même lorsque la position des capteurs est unique dans l'espace (plan ou en dimension 3), ces algorithmes peuvent trouver une configuration en dimension supérieure. Pour éviter cet effet indésirable, la rigidité universelle, une notion plus forte que la rigidité globale est requise. Une charpente est universellement rigide si elle est exclusivement déterminée modulo une congruence dans tous les espaces. Caractériser combinatoirement la rigidité universelle des charpentes génériques semble plus difficile que la rigidité globale, en partie car elle ne dépend pas uniquement du graphe sous-jacent, même lorsque les sommets sont dans une position générique. En fait, même en dimension 1, aucune caractérisation combinatoire de la rigidité universelle est connue.

1.1 Frameworks and rigidity

Rigidity theory originally concerns with the rigidity/flexibility of frameworks. A framework is a mathematical model describing a real structure. An early source of structures that provoked the study of rigidity is from architectural engineering. Consider a planar structure constituting of solid bars that are joined at extremities, called joints, by junctions that allow two joined bars to move freely in the plane about this joint. A natural question is whether the shape of the structure can be deformed. For example, the rectangular planar structure in Figure 1.1 can be deformed continuously to a non-rectangular one as shown by the dashed figure. We say that it is flexible. Meanwhile, the planar structure in Figure 1.2 can not be deformed continuously to another shape without being ripped apart. We say that it is rigid, or more precisely, locally rigid.

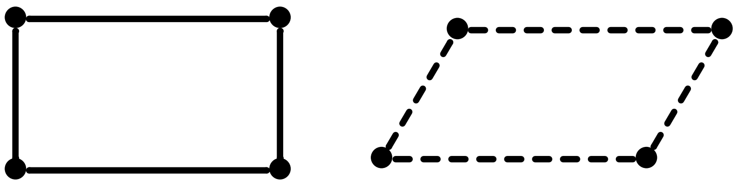


Figure 1.1: A flexible bar-and-joint structure in the plane.

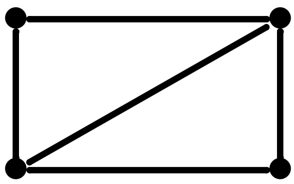


Figure 1.2: A rigid bar-and-joint structure in the plane.

Such a planar bar-and-joint structure can be modeled by a pair $(G, \mathbf{p})$ of a graph $G = (V, E)$ and a map $\mathbf{p} : V \to \mathbb{R}^2$. The vertices V of G correspond to the joints and the edges E of G correspond to the bars. The map $\mathbf{p}$ determines the location of the joints. In the same way, a bar-and-joint structure in 3-space (and more generally in dimension d) can be modeled with a map $\mathbf{p} : V \to \mathbb{R}^3$ (and $\mathbf{p} : V \to \mathbb{R}^d$, respectively). We call this pair a bar-joint framework. A continuous motion of a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is a continuous motion of the vertices of G in $\mathbb{R}^d$ which preserves the edge lengths. A framework is said to be locally rigid if every continuous motion of $(G, \mathbf{p})$ preserves also the pairwise distance between all

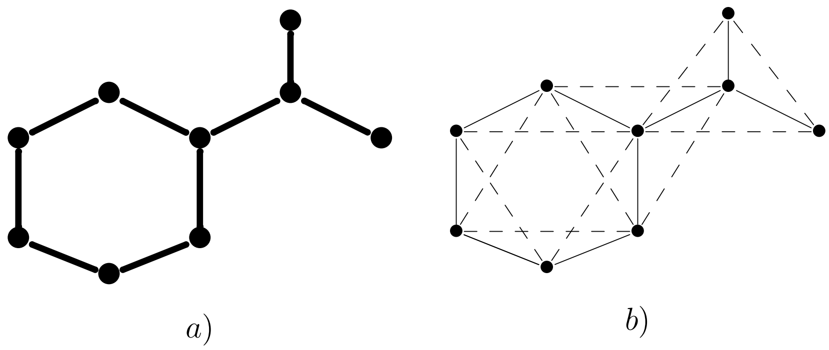


Figure 1.3: a) A molecule structure with atoms and bonds.
b) The corresponding bar-joint framework.

vertices of G. Equivalently, every continuous motion of $(G, \mathbf{p})$ is the restriction of a congruent motion of the whole space.

The problem of determining the local rigidity of a framework is hard in general. One way to deal with this is to consider the linearized problem where we focus on the instantaneous motion of the joints. An infinitesimal motion of a framework is an assignment of a vector $\mu(v)$ to each vertex $v \in V$ ($\mu(v)$ can be viewed as the instantaneous velocity of the joint) such that for each edge (i.e., bar) the instantaneous displacements along this edge induced by this infinitesimal motion at two ends are the same, which means

$$(p(u) - p(v))(\mu(u) - \mu(v)) = 0 \quad \text{for all } uv \in E. \tag{1.1.1}$$

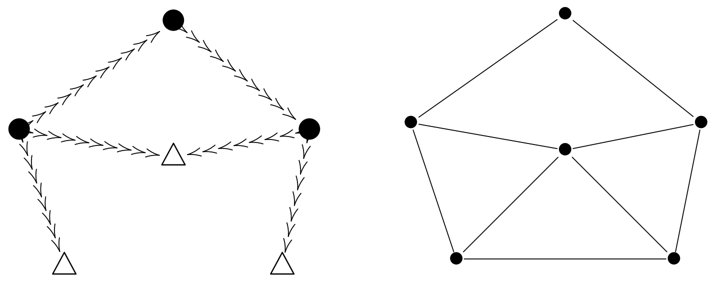
(See Figure 3.2.) A framework is said to be infinitesimally rigid if all its infinitesimal motions can be obtained as the instantaneous velocity vectors of a restriction of a congruent motion of the whole space on V. As a matter of fact, infinitesimal rigidity is a stronger property than local rigidity, but it is a good alternative for local rigidity since it turns out that they coincide in "most" cases, namely for frameworks in generic position. Infinitesimal rigidity is more tractable than local rigidity since it can be decided by calculating the rank of a matrix derived from the linear system (1.1.1) for μ.

The study of local rigidity/flexibility has found important applications in predicting flexibility of protein molecules. We can regard a molecular structure with covalent bonds as a bar-joint framework in 3-space: each atom is a joint and each covalent bond works as a bar that fixes the distance between two atoms. Since the angles between convalent bonds of an atom are also fixed, we need to add one more edge between each pair of neighbors of an atom (Figure 1.3). As protein molecules are often very large with thousands of atoms and bonds, the linear algebra approach by calculating the rank of the rigidity matrix becomes non efficient. The study of large frameworks necessitates the combinatorial results on the underlying graphs of these frameworks which would facilitate the design of efficient algorithms. The fundamental result by Laman (1970) asserts that the local/infinitesimal rigidity of a generic framework in the plane can be discerned by a simple counting condition on vertices and edges. This counting condition is based on the following intuitive reasoning: Every motion of a point in the plane is a combination of a horizontal and a vertical motion. So we say that a point has 2 degrees of freedom. If a framework on n vertices has no edge then it has $2n$ degrees of freedom. Adding an edge to the framework reduces its degrees of freedom by at most 1. However, for $n \geq 2$, there are always motions that can not be blocked by edges, they are congruent motions of the framework. We can count for these 3 independent motions: 2 for translations and 1 for rotation. Therefore, in an infinitesimally rigid framework without redundant edges one should expect that

1. the framework has $2n - 3$ edges in total, and
2. in each subframework with n' vertices, there are at most $2n' - 3$ edges.

The necessity of these conditions has been known since James Clerk Maxwell's time [81]. Laman showed that, for generic frameworks, it is also sufficient. In dimension 3 and higher, up to now, no combinatorial characterization is known for generic rigidity. Nevertheless, for the special class of frameworks that describe molecular structures, a long-conjectured combinatorial characterization, known under the name of Molecular Conjecture (by Tay and Whiteley 1984), has been proven to be true. Softwares based on this characterization such as FIRST, FRODA,... ([67],[105]) had been developed even before the confirmation of its mathematical correctness.

Let us move to other notions of rigidity. The first one concerns with the global uniqueness of frameworks, namely, if the set of distance constraints imposed by bars is sufficient to determine the framework up to congruence. As an example, let us consider the sensor network localization problem. In a sensor network, autonomous sensors collect information on enviroment condition such as temperatures, pressures, chemical agents,... and send data through the network to a center. Though not all sensors are equipped with GPS receivers, which are costly and battery consuming, to localize directly their position, the distance between some pairs of sensors can be calculated (using radio signals for close sensors, for example). A



(a) (b)

Figure 1.4: (a) A wireless sensor network. Elements with GPS receivers are represented with triangles and the others with black circles. The arrow lines represent the communicability between sensors.
(b) The corresponding bar-joint framework.

well-designed sensor network should allow one to locate all sensors based on this information. Such a network can be converted to a bar-joint framework by putting edges between pairs of sensors whose distances are known as well as between all sensors possessing a GPS receiver (Figure 1.4). It is evident that the first condition for the localizability is that the location of each sensor is uniquely determined by the available information on the distances. It is equivalent to the global rigidity of the corresponding framework.

Although global rigidity looks quite different from local rigidity, they are actually tightly related. It is easy to see that global rigidity implies local rigidity. In the other direction, it is shown that a planar generic framework is globally rigid if and only if one must remove at least 3 vertices to disconnect it and it remains locally rigid after the removal of any edge.

In practice, it is also important that the location of the sensors can be computed efficiently. To this end, semidefinite programming-based algorithms are proposed. However, even when the location of the sensors is unique in the considered space (plane or 3-space), these algorithms may find a configuration in a higher dimension. To prevent this undesirable outcome, universal rigidity, a stronger notion than global rigidity is of order. A framework is universally rigid if it is uniquely determined up to congruence in any space. Characterizing combinatorially universal rigidity of generic frameworks seems more difficult than global rigidity, partly due to the fact that it does not depend uniquely on the underlying graphs, even when the vertices are in generic position (Figure 1.5). As a matter of fact, even in

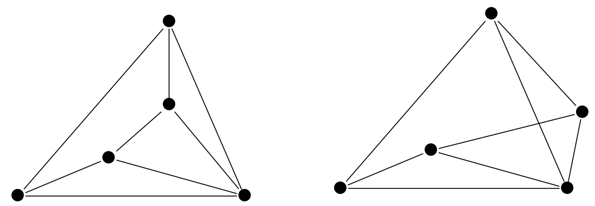


Figure 1.5: The planar generic framework on the left is universally rigid while the one on the right is not.

dimension 1, no combinatorial characterization of universal rigidity is known.

Driven by practical applications, many types of frameworks are extensively studied. Rigidity properties in these models are defined in the same manner as for bar-joint frameworks: local/infinitesimal rigidity for the uniqueness under continuous/infinitesimal motions, global rigidity for the uniqueness in the same dimension and universal rigidity for the uniqueness in all dimensions. We brief here some models studied in our thesis.

1. Direction-length frameworks: This is an extended model of bar-joint frameworks where beside distance constraints we have also direction constraints between vertices.

2. Body-bar frameworks and body-bar frameworks with boundaries. In a body-bar framework, bars link solid bodies and in a body-bar framework with boundaries, some bodies are linked to a fixed external environment by bars or they are simply pinned down. These models can be converted to bar-joint model but are interesting in their own right as combinatorial characterizations for rigidity in these models are obtained in all dimensions.

3. Body-length-direction frameworks: These are extended models of body-bar frameworks where we have both distance and direction constraints between bodies. We also consider different types of bodies which allow different types of motions.

The study of these frameworks may have applications in Computer-Aided-Design or in sensor network localization.

4. Tensegrity frameworks are a rather different model. In these frameworks we have three types of edges corresponding to three types of constraints: bars which fix distances between pairs of vertices, cables which prevent distances

from increasing and struts which prevent distances from decreasing. The consideration of these frameworks arose naturally in architectural engineering since materials which resist well compression are not strong in withstanding tension and vice versa. Using cable and strut-like elements helps reduce the cost and mass of the structure. Recent researches also propose considering cell structure as a tensegrity structure [56].

1.2 Our contributions and organization of the thesis

Chapter 2 reviews basic notions and facts used in our thesis, especially on graphs and digraphs, combinatorial optimization tools (matroid theory, submodular function,...) as well as algebra and linear algebra. Chapter 3 gives a basic formal introduction to the rigidity theory of bar-joint frameworks. The main results of this thesis appear from Chapter 4 to Chapter 8. Below, we give a summary of these chapters.

Chapter 4: Matroid approach. This chapter presents our early approach to the problem of characterizing local/infinitesimal rigidity in generic frameworks. Viewing this problem as characterizing the generic rigidity matroid, we study abstract rigidity matroids, a generalization of the generic rigidity matroid. We solve an open question by Graver, Servatius and Servatius [47] on the characterization of abstract rigidity matroids in dimension 2 and extend the result to all dimensions. We also introduce the notion of 1-extendable abstract rigidity matroids and investigate the relation between these matroids and the generic rigidity matroids. These results compose an article appeared in SIDMA [85]. Furthermore, inspired by the counting condition by Laman, we propose the study of intersecting submodular functions that induce abstract rigidity matroids and obtain a characterization for these functions.

The core of our thesis lies in constructive approaches to the problem of rigidity. By "constructive approach" we first mean an approach that focuses on inductive constructions of rigid frameworks or graphs and the effect of extension operations on frameworks or graphs. This approach has proved its power particularly in characterizing infinitesimal/local rigidity (see, e.g., [101, 99, 98, 94, 72]). It is also sucessfully employed in characterizing global rigidity in [58, 24]. Second, an alternative to inductive construction is to find a decomposition for underlying graph of (minimally) rigid frameworks. Such a decomposition can be used to construct an explicit rigid framework. This approach often results in a simpler proof for the desired characterization. (See [100, 107, 60] for examples.)

Chapter 5: Inductive construction and decomposition of graphs. First, in Section 5.2, we provide an inductive construction and a decomposition for graded sparse graphs. These graphs arise when one considers frameworks with mixed constraints, then different types of edges are subject to different sparsity conditions. We also study the graded sparse matroids determined by these graphs, obtain the rank formula as well as a decomposition of these matroids. The result in this section is from a joint work with Bill Jackson [66].

Section 5.3 considers another notion of sparse graphs: $(\mathbf{b}, l)$-sparse graphs, where the sparsity of subgraphs depend on their vertices. The motivation for this notion comes from the consideration of frameworks with bodies of different dimensions. We give an inductive construction for these graphs and also a characterization of these graphs as resulting graphs in a pebble game.

Our results on these different types of sparse graphs generalize the classic result of Nash-Williams [83] on the decomposition of graphs into edge-disjoint spanning trees. The dual result of this is about packing of edge-disjoint spanning trees by Tutte [104] and Nash-Williams [82]. Their results can be obtained easily from the directed counterpart on packing of arc-disjoint arborescences by Edmonds [31] via an orientation result by Frank.

Our third contribution in Chapter 5 is a generalization of the result of Edmonds to packing of arborescences whose roots are constrained to some matroidal condition (Section 5.4). Using a general orientation result by Frank [38], we obtain a short proof for a result of Katoh and Tanigawa which generalizes Tutte and Nash-Williams' result for characterizing rigidity of frameworks with boundaries. This is the result of a joint work with Olivier Durand de Gevigney and Zoltán Szigeti which appeared in SIDMA [28].

Chapter 6: Infinitesimal rigidity. This chapter focuses on the infinitesimal rigidity of several types of frameworks with mixed constraints. In Section 6.3, we obtain combinatorial characterizations of infinitesimal rigidity in generic body-length-direction frameworks, using the decomposition for graded-sparse graphs in Chapter 5. This part is also from the previously mentioned joint work with Bill Jackson.

Section 6.4 discusses extension operations for direction-length frameworks in general dimension. We extend the definition of 0-extension and 1-extension in $\mathbb{R}^2$

to d-dimension case and investigate the effect of these extension operations on the infinitesimal rigidity and boundedness of generic frameworks.

Chapter 7: Global rigidity of direction-length frameworks. We extend a result of Jackson and Jordán on the global rigidity preservingness of 1-extensions on generic direction-length frameworks in $\mathbb{R}^2$ to all dimensions. This chapter consists of an article appeared in IJCGA [86].

Chapter 8: Universal rigidity. While local/infinitesimal rigidity and global rigidity are well studied with abundant results, especially for generic frameworks in dimension 1 and 2, little is known about universal rigidity even in dimension 1. We offer the study of universal rigidity in two directions.

First, in Section 8.2, we explore universal rigidity on the line. We obtain a complete characterization for universal rigidity of complete bipartite frameworks on the line and show that the only generically universally rigid bipartite graph in $\mathbb{R}^d$ is the single edge $K_{1,1}$ for all $d \geq 1$. Many open questions and conjectures inspired by our result are discussed. This section consists of a joint article with Tibor Jordán [70].

The second direction is to relax the condition on the genericity of frameworks. In Section 8.3, we strengthen earlier results on sufficient condition for universal rigidity of bar-joint frameworks [8] and tensegrity frameworks [19] to allow configurations in non-general position. This is the result from a joint work with Abdo Alfakih [6].

Chapter 2

Preliminaries

Contents

2.1 Basic notions

We will use $\mathbb{Z}$ and $\mathbb{Z}_+$ to denote the set of integers and non negative integers, respectively. The set of rational numbers and real numbers are denoted by $\mathbb{Q}$ and $\mathbb{R}$ respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbf{0}$ denotes a zero matrix of appropriate dimensions or the origin of an Euclidean space. The Euclidean norm of a vector/point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is denote by $\|x\|$, that is

$$\|x\|^2 = x_1^2 + \cdots + x_n^2.$$

The number of 2-element subsets of an n-element set is denoted by $\binom{n}{2}$.

For a finite set S the number of elements of S is denoted by $|S|$. The collection of all the subsets of S is denoted by 2^S. If a set A is a subset of a set B then we write $A \subseteq B$. We also write $A \subset B$ when $A \subseteq B$ and $A \neq B$ and say that A is a proper subset of B. A *multiset* is a generalization of the notion of set, where each member may appear more than once. For two sets A, B, let $A \setminus B$ or $A - B$ denote the set of all elements belonging to A but not to B. $A \cup B$ and $A \cap B$ denote the union and intersection of A and B respectively. We also write $A - x$ for $A \setminus \{x\}$ and $A + x$ or $A \cup x$ for $A \cup \{x\}$.

A partition of a non-empty set S is a collection $\mathcal{P} = \{P_1, \dots, P_n\}$ where $P_1, \dots, P_n$ are non-empty pairwise disjoint subsets of S such that $P_1 \cup \cdots \cup P_n = S$.

Let $f : X \to Y$ be a map, $A \subseteq X$, and $B \subseteq Y$. The *image* of A by f is $f(A) = \{f(x) : x \in A\}$ and the *pre-image* of B by f is $f^{-1}(B) = \{x \in A : f(x) \in B\}$.

The *Jacobian matrix* of a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$, at a point p is the $m \times n$ matrix

$$\mathrm{d}f|_p = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

2.2 Graphs and digraphs

A *graph* (or undirected graph) is a pair $G = (V, E)$ where V is a non-empty finite set and E consists of 2-element multisubsets of V. The elements of V are called *vertices* or *nodes* of G and the elements of E are called the *edges* of G. Sometimes, to be more precise, we use $V(G)$ and $E(G)$ to refer to the vertex set and the edge set of G. An edge $\{u, v\}$ of G is often denoted by uv and u, v are called the *extremities* or *ends* of the edge uv. We say that the edge uv is *incident* to u and v. If $u = v$ then the edge uv is called a *loop*. A graph is often depicted by a set of dots for the vertices and line segments or curves for the edges. A graph without loops and multiple edges is called a *simple graph*. The *degree* of a verter v in a graph G, denoted by $d(v)$, is the number of non-loop edges incident to v plus twice the number of loops at v.

Two simple graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if there exists a bijection $\phi : V \to V'$ such that for every $u, v \in V$, uv is an edge of G if and only if $\phi(u)\phi(v)$ is an edge of G'.

A *complete graph* is a simple graph where between each pair of vertices there is an edge. A complete graph on n vertices is often denoted by K_n. We also use $K(V)$ to denote the complete graph on a vertex set V. A *bipartite graph* is a graph where the vertex set can be partitioned into two sets X and Y such that there is no edge between two vertices in X and there is no edge between two vertices in Y. A *complete bipartite graph* is a bipartite graph with a vertex partition X, Y such that for each pair of vertices $x \in X$ and $y \in Y$, xy is an edge. If $|X| = m$ and $|Y| = n$, the complete bipartite graph with the vertex partition X, Y is denoted by $K_{m,n}$.

Let $G = (V, E)$ be a graph and X, Y be subsets of V. Throughout this thesis, we will use $E(X)$ to denote the set of edges induced by X in G, i.e., those edges of G with both extremities in X, and $i(X)$ to denote $|E(X)|$. If $X = \{v\}$, $i(X) = i(v)$ is simply the number of loops at v. We will use $E(X, Y)$ to denote the set of edges from X to Y, i.e., those edges of G with one extremity in X and the other in Y. We denote $|E(X, Y)|$ by $i(X, Y)$. A *subgraph* of $G = (V, E)$ is a graph $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E(V')$. It is called an *induced subgraph* of G if $E' = E(V')$. The vertex set of an edge set F is denoted by $V(F)$.

A *path* from a vertex u to a vertex v in a graph $G = (V, E)$ is a sequence $u = v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k = v$ of vertices $v_0, \ldots, v_k$ in V and edges $e_1, \ldots, e_k$ in E such that v_{i-1}, v_i are two ends of e_i for $i = 1, \ldots, k$. We often identify a path with its sequence of edges. The edge set may be empty if $u = v$. If there is a path from u to v in G then we say that v is *reachable* from u.

A *connected graph* is a graph $G = (V, E)$ whose vertex set V can not be partitioned into two nonempty subsets X and Y such that $i(X, Y) = 0$. Equivalently, a connected graph is a graph G such that, for each pair of vertices u, v, there is a path from u to v in G. A *connected component* of a graph G is a maximal connected subgraph of G. Obviously, connected components of G are vertex-disjoint. An edge e of G is called a *cut-edge* if the deletion of e from G increases its number of connected component.

A *cycle* is a connected graph in which each vertex has degree 2. If an edge of G is not a cut-edge then it must belong to a cycle in G. A *forest* is a graph that does not contain any cycles as its subgraphs. A *tree* is a connected forest. A subgraph H of G is said to be *spanning* if $V(H) = V(G)$. A tree is a *spanning tree* of G if it is a spanning subgraph of G.

Let k be a positive integer. A graph G is said to be *k-vertex-connected*, or simply *k-connected*, if $|V(G)| > k$ and one has to remove at least k vertices to disconnect the graph. Note that 1-connected graphs are simply connected graphs.

Let $\mathcal{P}$ be a partition of the vertex set V of G into non-empty subsets. We denote by $E_G(\mathcal{P})$ the set of all edges in $E(X, Y)$ for every $X, Y \in \mathcal{P}, X \neq Y$. Edges in $E_G(\mathcal{P})$ are called *crossing edges* of the partition $\mathcal{P}$. We often denote $|E_G(\mathcal{P})|$ by $e_G(\mathcal{P})$.

A *directed graph* (or a *digraph*) is a pair $D = (V, A)$ where V is a finite set and A consists of ordered pairs of elements of V (where we can have pairs of the same element). The elements of V are called the vertices of D and the element of A are called the *arcs* of D. An arc (u, v) is often denoted by uv; u is called the *tail* and v is called the *head* of the arc uv. If $u = v$ then the arc uv is called a loop. A digraph is depicted in the same way as an undirected graph except that we use an arrow to denote the direction from u to v of an arc uv. An arc uv from u to v is said to be an *out-arc* of u and an *in-arc* of v.

Let $D = (V, A)$ be a digraph and X a subset of V. An arc uv with $u \in V \setminus X$ and $v \in X$ is said to be an *entering arc* of X. An *outgoing arc* of X is an entering arc of $V \setminus X$. The set of all arcs of D entering X is denoted by $R^-_D(X)$, the number of its elements is denoted by $\rho_D(X)$. When $X = \{v\}$, we simply write $R^-_D(v)$ and $\rho_D(v)$. Concerning the outgoing arc of X we will use $R^+_D(X)$, $\delta_D(X)$ respectively.

A *dipath* from a vertex u to a vertex v in a digraph $D = (V, A)$ is a sequence $u = v_0, a_1, v_1, a_2, \ldots, v_{n-1}, a_n, v_n = v$ of vertices $v_0, \ldots, v_n$ in V and arcs $a_1, \ldots, a_n$ in A such that v_{i-1} is the tail and v_i is the head of the arc a_i for $i = 1, \ldots, n$. We often identify a dipath with the sequence of its arcs. If there exists a dipath from v to u in D then we say that u is *reachable* from v in D.

We say that D is an *r-arborescence* if D is a directed tree, r is a vertex of D of in-degree 0 and all the other vertices of D are of in-degree 1. We note that an r-arborescence may consist of only the vertex r and no arc. Note also that an r-arborescence has a unique vertex of in-degree 0, namely r. We also use *out-arborescence* to refer to an arborescence when we want to distinguish it with an *in-arborescence*, which is a directed graph where one vertex has out-degree 0 and all the other vertices have out-degree 1.

A sub-digraph H of D is called *spanning* if its vertex set $V(H)$ coincides with V. A digraph D is *strongly connected* if for every two vertices u, v of D, u is reachable from v and vice versa. D is said to be k-connected if $|V(D)| > k$ and one need to remove at least k vertices to make D non strongly connected.

2.3 Matroid theory

Definition 2.3.1 (Matroid)**.** *A matroid $\mathcal{M}$ is a pair $(S, \mathcal{I})$ of a finite set S and a collection $\mathcal{I}$ of subsets of S that satisfies the following independence axioms.*

(I0) $\emptyset \in \mathcal{I}$.

(I1) *If $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$.*

(I2) *If $I, J \in \mathcal{I}$ and $|I| > |J|$ then there exists $x \in I \setminus J$ such that $J \cup x \in \mathcal{I}$.*

Matroid is a structure that generalizes the notion of linear independence. If S is a finite set of vectors in a vector space, the collection $\mathcal{I}$ of linearly independent subsets of S verifies the three properties above. We call this matroid the *linear matroid* defined by S. Matroid theory draws also motivations from graph theory. Given a graph $G = (V, E)$, the edge sets of all subgraphs of G that are forests form the independent sets of a matroid on E. We call this matroid the *graphic matroid* of G. In fact, this matroid is also a linear matroid.

The set S is called the *ground set* of the matroid $\mathcal{M}$ and the sets in $\mathcal{I}$ are called the *independent sets*. Sets that are not independent are said to be *dependent*. Minimal dependent sets are called *circuits*. If $\{x\}$ is a circuit then x is called a *loop*. Two elements x, y of a matroid $\mathcal{M}$ are said to be *parallel* if $\{x, y\}$ is a circuit of $\mathcal{M}$.

A *base* of a matroid $\mathcal{M}$ is a maximal inclusionwise independent set. The collection $\mathcal{B}$ of bases of a matroid $\mathcal{M}$ satisfies the following properties.

(B0) $\mathcal{B}$ is not empty.

(B1) If B_1 and B_2 are in $\mathcal{B}$ then $|B_1| = |B_2|$.

(B2) (*Exchange axiom*) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ then there exists $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{B}$.

These axioms on bases can be used to define a matroid. In fact, if a collection $\mathcal{B}$ of subsets of a finite set S verifies the axioms (B0), (B1), (B2) then it determines uniquely a matroid by defining the independent sets as all the subsets of bases. Furthermore, if we set

$$\mathcal{B}^* = \{S \setminus B : B \in \mathcal{B}\}$$

then $\mathcal{B}^*$ also satisfies (B0), (B1), (B2) and therefore determines a matroid on S. We call this matroid the *dual matroid* of $\mathcal{M}$. Bases and circuits of the dual matroid of $\mathcal{M}$ are called *cobases* and *cocircuits* of $\mathcal{M}$. The complement of a cocircuit is called a *hyperplane* of $\mathcal{M}$.

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. The *rank* of a set $X \subseteq S$ is the maximum cardinality of an independent subset of X:

$$r_{\mathcal{M}}(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}\}.$$

When the matroid is clear from the context, we may write $r(X)$ instead of $r_{\mathcal{M}}(X)$. The rank of a matroid satisfies the following properties.

(R1) $0 \leq r(X) \leq |X|$ for every $X \subseteq S$.

(R2) If $X \subseteq Y$ then $r(X) \leq r(Y)$ for every $X, Y \subseteq S$.

(R3) (*Submodularity*) If $X, Y \subseteq S$ then $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

A matroid is also determined uniquely by its rank function, i.e, if $r : 2^S \to \mathbb{Z}^+$ satisfies the three axioms (R1), (R2) and (R3) then it defines a matroid on S: the independent sets are subsets X of S with $r(X) = |X|$.

The *closure operator* $\mathrm{cl}_{\mathcal{M}}(\cdot)$ of a matroid $\mathcal{M}$ on S is an operator on the collection of subsets of S defined by

$$\mathrm{cl}_{\mathcal{M}}(X) = \{x \in S : r(X \cup x) = r(X)\}.$$

for every $X \subseteq S$. We may write $\mathrm{cl}(X)$ when $\mathcal{M}$ is clear from the context.

Given a matroid $\mathcal{M} = (S, \mathcal{I})$ on a finite set S and a subset S' of S, we can obtain a matroid on S' by defining

$$\mathcal{I}' = \{I \in \mathcal{I} : I \subseteq S'\}.$$

It is easy to see that $\mathcal{I}' = \{I \cap S' : I \in \mathcal{I}\}$. The fact that $\mathcal{I}'$ verifies (I0), (I1) and (I2) follows from that of $\mathcal{I}$. Hence, $\mathcal{I}'$ forms the independent sets of a matroid on S'. We call this matroid $\mathcal{M}' = (S, \mathcal{I}')$ the *restriction* of $\mathcal{M}$ on S'.

A *free matroid* on a finite set S is the matroid on S whose independent sets are all the subsets of S.

Count matroids and sparse graphs

Let us consider a class of matroids that plays an important role in combinarial rigidity theory: count matroids.

Let $G = (V, E)$ be a graph and $\mathbf{b} : V \to \mathbb{Z}_+$. Here we use the bold symbol to emphasize the fact that $\mathbf{b}$ is a map. For the image of $v \in V$ and $X \subseteq V$ we write $b(v), b(X)$ respectively. Let $\mathrm{b}_{\min}$ denote $\min\{b(v) : v \in V\}$. Suppose that l is an integer with $0 \leq l < 2\mathrm{b}_{\min}$ for every $uv \in E$. Then it is not difficult to show that the collection

$$\mathcal{I} = \{F \subseteq E : i_F(X) \leq b(X) - l \quad \text{for every } X \subseteq V \text{ with } i_F(X) > 0\}$$

where $i_F(X)$ denotes the number of edges in F with both extremities belonging to X, forms the independent sets of a matroid on E (see, e.g., [39, Theorem 13.5.1] for a proof). Note that, by our assumption $0 \leq l < 2\mathrm{b}_{\min}$, the condition for the independence of an edge set F implies that if $|X| \geq 2$ then $i_F(X) \leq b(X) - l$. The matroid $(E, \mathcal{I})$ is called a $(\mathbf{b}, l)$*-count matroid* on G. If $b(v) = 1$ for all $v \in V$ and $l = 1$, the $(\mathbf{b}, l)$-count matroid on G coincides with the graphic matroid on G. When $b(v) = 2$ for all $v \in V$ and $l = 3$, the $(\mathbf{b}, l)$-count matroid coincides with the 2-dimensional generic rigidity matroid (more details in Chapter 3).

A graph G with the edge set independent in a $(\mathbf{b}, l)$-count matroid on G is called a $(\mathbf{b}, l)$*-sparse graph*. A $(\mathbf{b}, l)$*-tight graph* is a $(\mathbf{b}, l)$-sparse graph $G = (V, E)$ with $|E| = b(V) - l$. When all $b(v)$ take the same value m, we also refer to these graphs as (m, l)*-sparse graphs* and (m, l)*-tight graphs*, respectively. In some context, to emphasize the sparseness, we also say *tight sparse graphs*. Familiar examples are $(1, 1)$-sparse graphs which are actually forests, and $(1, 1)$-tight graphs which are trees.

Matroid union

One way to get a matroid from other matroids is taking their unions. Let $\mathcal{M}_1 = (S_1, \mathcal{I}_1), \ldots, \mathcal{M}_k = (S_k, \mathcal{I}_k)$ be k matroids. The *matroid union* of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ is the matroid $\mathcal{M} = (S, \mathcal{I})$ where $S = S_1 \cup \cdots \cup S_k$ and

$$\mathcal{I} = \{I_1 \cup \cdots \cup I_k : I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\}.$$

If $S_i \neq S$, we can extend the matroid $\mathcal{M}_i$ to a matroid on S by defining all elements of $S \setminus S_i$ to be loops. We identify this new matroid with $\mathcal{M}_i$. Therefore, for the sake of convenience, we may suppose that $\mathcal{M}_i$ are matroids on the same ground set S.

Many fundamental results become clear when viewed under the matroid union angle. For instance, the result of Nash-Williams that (k, k)-sparse graphs are the union of k edge-disjoint forests can be stated as every (k, k)-sparse matroid is the union of k $(1, 1)$-sparse matroids (more details in Chapter 5).

The rank function for the matroid union is determined as in the following result. (For a proof, see e.g. [87, Theorem 12.3.1].)

Theorem 2.3.1. *Let $\mathcal{M}$ be the matroid union of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ with rank functions $r_{\mathcal{M}_1}, \ldots, r_{\mathcal{M}_k}$. Then the rank function of $\mathcal{M}$ is given by*

$$r_{\mathcal{M}}(X) = \min\{r_{\mathcal{M}_1}(Y) + \cdots + r_{\mathcal{M}_k}(Y) + |X \setminus Y| : Y \subseteq X\} \tag{2.3.1}$$

for every $X \subseteq S$.

2.4 Submodular functions

Definition 2.4.1 (Submodular function)**.** *Let S be a finite set. An integer-valued function $f : 2^S \to \mathbb{Z}$ is said to be submodular if it satisfies*

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{2.4.1}$$

for every subset X, Y of S.

If $f : 2^S \to \mathbb{Z}$ satisfies inequality (2.4.1) for every $X, Y \subseteq S$ with $X \cap Y \neq \emptyset$ then it is called an *intersecting submodular function*. A function $g : 2^S \to \mathbb{Z}$ is a *supermodular function* (intersecting supermodular function, resp.) if $-g$ is submodular (intersecting submodular, resp.).

Submodular functions (and hence supermodular functions) play an important role in combinatorial optimization because of its omnipresence. For example in a graph G, $\rho(X)$ is a submodular function, $i(X)$ is a supermodular function, for $X \subseteq V(G)$, and $m|V(F)| - l$ is an intersecting submodular function, for $F \subseteq E(G)$, where $1 \leq m < 2l$. Moreover, submodular functions are of special interest in combinatorial optimization as they can be minimized in polynomial time.

Theorem 2.4.1 (Iwata, Fleischer and Fujishige [57], Schrijver [92]). *Given a submodular function $f : 2^S \to \mathbb{Z}$, a set $U \subseteq S$ that minimizes $f(U)$ can be found in polynomial time.*

Submodular functions are closely related to matroids. As mentioned above, the rank of a matroid is a submodular function. Conversely, given a nondecreasing intersecting submodular function $f : 2^S \to \mathbb{Z}$, let us consider the collection

$$\mathcal{I}(f) = \{I \subseteq S : |J| \leq f(J) \quad \text{for every } J \subseteq I, J \neq \emptyset\}.$$

Theorem 2.4.2. *$\mathcal{I}(f)$ forms the independent sets of a matroid $M(f)$ on S. The rank function of $\mathcal{M}(f)$ is determined by*

$$r_{\mathcal{M}(f)}(X) = \min\{|X_0| + \sum_{i=1}^{t} f(X_i) : \{X_0, X_1, \ldots, X_t\} \textit{ partitions } X\} \tag{2.4.2}$$

for $X \subseteq S$.

(For a proof, see [39], for example.)

The circuits of this matroid are minimal non empty sets C such that $f(C) < |C|$ and $f(C) \geq f(C - x) \geq |C| - 1$ for every x in C. Therefore, we have the following.

Proposition 2.4.3. *If C is a circuit of the matroid $\mathcal{M}(f)$ induced by a nondecreasing intersecting submodular function f then $f(C) = f(C - x) = |C| - 1$.*

2.5 Algebra and linear algebra

Let K be a subfield of the complex field $\mathbb{C}$. By $K[X_1, \ldots, X_n]$ we denote the ring of all n-variable polynomials $f[X_1, \ldots, X_n]$ with coefficients in K.

The *extension field* of a field $K \subseteq \mathbb{C}$ by $p_1, \ldots, p_n \in \mathbb{C}$, denoted by $K(p_1, \ldots, p_n)$, is the smallest subfield of $\mathbb{C}$ that contains $p_1, \ldots, p_n$. Equivalently, $K(p_1, \ldots, p_n) = \{f(p_1, \ldots, p_n)/g(p_1, \ldots, p_n) : f, g \in K[X_1, \ldots, X_n],\ g(p_1, \ldots, p_n) \neq 0\}$.

The *algebraic closure* of a field $K \subseteq \mathbb{C}$ is the smallest field $\overline{K} \subseteq \mathbb{C}$ such that every polynomial $f(X_1) \in \overline{K}[X_1]$ has a root in $\overline{K}$.

A set $\{p_1, \ldots, p_n\} \subset \mathbb{C}$ is said to be *algebraically independent* over K if there does not exist a polynominal $f(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$, non identical to zero, such that $f(p_1, \ldots, p_n) = 0$. It turns out that this algebraic independence is in fact matroidal.

Theorem 2.5.1 ([87] Theorem 6.7.1)**.** *Let L be an extension field of a field K and S a finite subset of L. Then the collection $\mathcal{I}$ of subsets of S that are algebraically independent over K is the set of the independent sets of a matroid on S.*

For an $m \times n$ matrix M we often use M_{ij} to denote the (i,j)-entry of M. The *transpose* of M is the $n \times m$ matrix denoted by M^T such that $M^T_{ij} = M_{ji}$.

We use I_n to denote the identity matrix of order n, that is the $n \times n$ matrix with all (i,i)-entries being 1 and all the other entries being 0. When the dimension n is clear from the context we simply write I for I_n. An $n \times n$ real matrix M is said to be *orthogonal* if $MM^T = M^T M = I_n$.

Sometimes we deal with matrices M whose entries are functions of a parameter t. We denote such a matrix by $M(t)$. The derivation $\dfrac{d}{dt}M(t)$ is simply the matrix whose (i,j)-entries are $\dfrac{d}{dt}M(t)_{ij}$. It is routine to verify that

$$\frac{d}{dt}\Big(M(t)N(t)\Big) = \Big(\frac{d}{dt}M(t)\Big)N(t) + M(t)\frac{d}{dt}N(t).$$

The nullspace $\operatorname{Ker} M$ of a matrix M with n columns is the linear space of all vector $x \in \mathbb{R}^n$ such that $Mx = 0$. For an $n \times n$ matrix M, let $\operatorname{trace}(M)$ denote the sum of all the diagonal entries of M. Let A be an $n \times m$ matrix and B an $m \times n$ matrix, then it is easy to verify that

$$\operatorname{trace}(AB) = \operatorname{trace}(BA).$$

An $n \times n$-matrix $A = (a_{ij})$ is *symmetric* if $a_{ij} = a_{ji}$ for all $1 \le i, j \le n$. The set of all real $n \times n$ matrices is denoted by $\mathcal{S}_n$. Then $\mathcal{S}_n$ is a vector space and we can define an inner product $\langle \cdot\, , \cdot \rangle$ in $\mathcal{S}_n$ by

$$\langle A\, , B \rangle = \operatorname{trace}(AB), \text{ for } A, B \in \mathcal{S}_n.$$

Let M be an $n \times n$ real matrix. A scalar λ is said to be an *eigenvalue* of M if there exists $x \in \mathbb{R}^n$, $x \neq \mathbf{0}$ such that $Mx = \lambda x$.

A symmetric $n \times n$ matrix A is said to be *positive definite* if for every non-zero $x \in \mathbb{R}^n$ we have $x^T Ax > 0$. A symmetric $n \times n$ matrix A is *positive semidefinite* (or PSD for short) if for every $x \in \mathbb{R}^n$ the inequality $x^T Ax \geq 0$ holds. PSD matrices are used to characterize the universal rigidity of frameworks (see Chapter 8). A *principal minor* of an $n \times n$ matrix A is the determinant of a square submatrix of A with rows and columns indexed by the same subset X of $\{1, \ldots, n\}$. If $X = \{1, \ldots, k\}$ for some $k \leq n$ then the principal minor is called a *leading principal minor*. The following lemma summarizes equivalent statements about the positive semidefiniteness of a matrix.

Lemma 2.5.2. *Let $A \in \mathcal{S}_n$. The following statements are equivalent.*

1. *A is a PSD matrix.*
2. *All eigenvalues of A are non-negative.*
3. *All principal minors of A are non-negative.*
4. *All leading principal minors of A are non-negative.*
5. *$A = XX^T$ for some $n \times n$ matrix X.*

A consequence of Lemma 2.5.2 is the following.

Lemma 2.5.3.

1. *If A is a PSD matrix in $\mathcal{S}_n$ then $\operatorname{trace}(A) \geq 0$ and $\operatorname{trace}(A) = 0$ if and only if $A = \mathbf{0}$.*
2. *Let A, B be two PSD matrices in $\mathcal{S}_n$, then $\langle A\,, B\rangle \geq 0$ and $\langle A\,, B\rangle = 0$ if and only if $AB = \mathbf{0}$.*

Proof. 1. By Lemma 2.5.2 we may assume that $A = XX^T$ for some $n \times n$ matrix $X = (x_{ij})$. Then

$$\operatorname{trace}(A) = \sum_{i,j} X_{ij}(X^T)_{ij} = \sum_{i,j} x_{ij}^2 \geq 0.$$

Moreover, $\operatorname{trace}(A) = 0$ if and only if $x_{ij} = 0$ for all i, j, which means that $X = 0$ and hence $A = 0$.

2. We may suppose that $A = XX^T$ and $B = YY^T$ for some $n \times n$ matrices X, Y. Then

$$\begin{aligned}\langle A\,, B\rangle = \operatorname{trace}(AB) = \operatorname{trace}(XX^TYY^T) &= \operatorname{trace}(Y^TXX^TY) \\ &= \operatorname{trace}((X^TY)^T(X^TY) \geq 0.\end{aligned}$$

Moreover, $\operatorname{trace}(AB) = 0$ if and only if the trace of the PSD matrix $(X^TY)^T(X^TY)$ is 0, which implies $X^TY = \mathbf{0}$ by the first statement and hence $AB = \mathbf{0}$ holds. □

A $n \times n$ matrix $A = (a_{ij})$ is *skew-symmetric* if $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq n$; in particular, $a_{ii} = 0$ for all $1 \leq i \leq n$. Skew-symmetric matrices will be used to describe infinitesimal rotations of rigid bodies.

Let x, y be two vectors in $\mathbb{R}^d$. The *join* of $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ is the $\binom{d}{2}$-dimensional vector

$$x \vee y = \left(\overset{(1,2)}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}}, \overset{(1,3)}{\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}}, \ \ldots \ , \overset{(i,j)}{\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}}, \ \ldots \ , \overset{(d-1,d)}{\begin{vmatrix} x_{d-1} & x_d \\ y_{d-1} & y_d \end{vmatrix}} \right)$$

Let $A = (a_{ij})$ be a $d \times d$ skew-symmetric matrix. We define a $\binom{d}{2}$-dimensional vector w associated with A by $w = (a_{12}, a_{13}, \ldots, a_{(d-1)d})$, i.e., by putting a_{ij} consecutively in the lexicographic order.

A related notion to join in $\mathbb{R}^3$ is the *cross product*

$$x \times y = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$$

which differs from $x \vee y$ in the order and a possible -1 scaling of entries.

The following equality relates inner product and join.

$$\langle Ay \,, x \rangle = \langle w \,, x \vee y \rangle. \tag{2.5.1}$$

Indeed,

$$\begin{aligned} \langle Ay \,, x \rangle &= \sum_{i,j} a_{ij} y_j x_i \\ &= \sum_{i<j} a_{ij} y_j x_i + \sum_{i>j} a_{ij} y_j x_i \quad (\text{since } a_{ii} = 0 \text{ for every } i) \\ &= \sum_{i<j} w_{ij} y_j x_i - \sum_{i<j} w_{ij} y_i x_j \\ &= \sum_{i<j} w_{ij} (x_i y_j - x_j y_i) \\ &= \langle w \,, x \vee y \rangle. \end{aligned}$$

2.6 Congruent motions and motions of rigid bodies

A *congruence* of $\mathbb{R}^d$ is a map $h : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\|h(p) - h(q)\| = \|p - q\| \quad \text{for all } p, q \in \mathbb{R}^d. \tag{2.6.1}$$

The following result is fundamental, a proof can be found in [77, 22].

Proposition 2.6.1. *A map $h : \mathbb{R}^d \to \mathbb{R}^d$ is a congruence if and only if there exists an orthogonal $d \times d$ matrix M such that*

$$h(p) = Mp + h(\mathbf{0}) \quad \textit{for all } p \in \mathbb{R}^d.$$

A *congruent motion* of $\mathbb{R}^d$ is a map $P : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ such that $P(t,p)$ is continuously differentiable in p and t and for each $t \in [0,1]$, the map $P(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a congruence, i.e.,

$$P(t,p) = M(t)p + P(t,\mathbf{0}) \quad \text{for all } t \in [0,1] \text{ and } p \in \mathbb{R}^d, \tag{2.6.2}$$

where $M(t)$ is a $d \times d$ orthogonal matrix.

Let $A(t) = \dfrac{d}{dt}M(t)$, differentiating both sides of the equality $M(t)M^T(t) = I$ we have,

$$M(t)A(t)^T + A(t)M(t)^T = \mathbf{0}.$$

At $t = 0$, $M(0) = I$, so we obtain $A(0)^T + A(0) = \mathbf{0}$, i.e., $A = A(0)$ is a skew-symmetric matrix.

By 2.6.2, the infinitesimal motion of p induced by the congruent motion P, i.e., the instantaneous velocity of p at instant $t = 0$, is given by

$$\frac{d}{dt}P(t,p)|_{t=0} = Ap + \mathbf{t}.$$

Here $\mathbf{t} = \dfrac{d}{dt}P(t,0)|_{t=0}$ which corresponds to the instantaneous translation, while A corresponds to the instantaneous rotation.

Inversely, suppose that A is a $d \times d$ skew-symmetric matrix and $\mathbf{t} \in \mathbb{R}^d$. Let

$$M(t) = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots \equiv e^{tA}.$$

Then it is easy to show that this series converges for every $t \in \mathbb{R}$, so $M(t)$ is well-defined. Moreover, $M(t)$ is infinitely differentiable and

$$\frac{d}{dt}M(t)|_{t=0} = A.$$

On the other hand, $M(t)M(t)^T = e^{tA}e^{tA^T} = e^{tA+tA^T} = I$, since A is skew-symmetric. Hence $M(t)$ is orthogonal and

$$P(t,p) = M(t)p + t\mathbf{t} \quad \text{for all } t \in [0,1] \text{ and } p \in \mathbb{R}^d,$$

is a congruent motion whose infinitesimal motion is $Ap + \mathbf{t}$ for all $p \in \mathbb{R}^d$.

Therefore, an infinitesimal motion of a congruent motion (or *infinitesimal congruence* for short) of $\mathbb{R}^d$ can be described by a pair $(A, \mathbf{t})$ of a skew-symmetric $d \times d$ matrix A and a vector $\mathbf{t} \in \mathbb{R}^d$. In particular, the space of infinitesimal congruences of $\mathbb{R}^d$ is of dimension $\binom{d+1}{2}$.

A *rigid body* is an idealization of a solid body in physical world. In a rigid body, the distance between any two points is constant. When there is no specification, a rigid body B in $\mathbb{R}^d$, or simply a body, is understood to be of full dimension, i.e, the set of points of B affinely spans $\mathbb{R}^d$

A motion of a rigid body $B \subset \mathbb{R}^d$ is a map $P : [0,1] \times B \to \mathbb{R}^d$ such that, for every $p \in B$, the map $P(\cdot, p)$ is continuous and

$$\|P(t,p) - P(t,q)\| = \|P(0,p) - P(0,q)\|, \quad \text{for every } t \in [0,1], \tag{2.6.3}$$

that is, the motion preserves the distance between points in the rigid body B, or equivalently speaking, $P(t, \cdot)$ is an isometry of B for every $t \in [0,1]$. In fact, $P(t,p)$ is the position of a point p in B at time t. These isometries $P(t, \cdot)$ can be extended uniquely to congruences of $\mathbb{R}^d$ [47, 22]

$$P(t,p) = M(t)p + P(t,0).$$

The above discussion shows that the infinitesimal motion of a rigid body in $\mathbb{R}^d$ can also be described by a pair $(A, \mathbf{t})$ of a skew-symmetric matrix A and a vector $\mathbf{t} \in \mathbb{R}^d$.

Chapter 3

Rigidity theory

Contents

3.1 Introduction

The theory of rigidity is characterized by its diversity from many aspects: motivations and applications, techniques, and, especially, models. It is thus difficult to cover the basic concepts in all prevailing models in this modest introduction into rigidity theory. In this chapter, we content ourselves with a description of the theory of rigidity for the bar-joint model. The first reason is that this model is easy to describe. Secondly, many other models can be converted to this model. Thirdly, ideas and techniques from the study of this model can be applied to other models as well. Lastly, although being quite simple to describe, the problems of characterizing the rigidity in the bar-joint model remains among the most difficult problems in rigidity theory. Therefore, bar-joint model serves as a base model to understand basic concepts, techniques as well as challenges in rigidity theory.

A *d-dimensional bar-joint framework* is a pair $(G, \mathbf{p})$ of a simple graph $G = (V, E)$ and an *embedding* $\mathbf{p}$ which maps each vertex $v \in V$ to a point $p(v)$ in $\mathbb{R}^d$. The vertices and the edges of G model the joints and the bars of a bar-and-joint structure while $\mathbf{p}$ is the placement of the structure in the d-dimensional Euclidean space. The pair $(G, \mathbf{p})$ is also called a d-dimensional bar-joint *realization* of G and sometimes we refer to $\mathbf{p}$ as a *configuration* of V in $\mathbb{R}^d$. The *affine dimension* of a configuration $\mathbf{p}$ is the dimension of the affine space spanned by $\{p(v) : v \in V\}$. We say that a configuration $\mathbf{p}$ or a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is of *full dimension* if the affine dimension of $\mathbf{p}$ is d. The embedding $\mathbf{p}$ can also be regarded as a point $\mathbf{p} \in \mathbb{R}^{d|V|}$.

Two realizations $(G, \mathbf{p})$ and $(G, \mathbf{q})$ of G in $\mathbb{R}^d$ are said to be *equivalent* if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\|, \quad \text{for every edge } uv \text{ in } E,$$

i.e., the length of the bars are the same in these two realizations. They are said to be *congruent* if

$$\|p(u) - p(v)\| = \|q(u) - q(v)\|, \quad \text{for every pair of vertices } u, v \text{ in } V.$$

Definition 3.1.1 (Rigidity map). *The d-dimensional rigidity map of a graph G is the map $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ defined by*

$$f_G(\mathbf{p}) = (\dots, \overset{uv\in E}{\|p(u) - p(v)\|^2}, \dots)^T$$

Note that two frameworks $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent if and only if $f_G(\mathbf{p}) = f_G(\mathbf{q})$. They are congruent if $f_{K(V)}(\mathbf{p}) = f_{K(V)}(\mathbf{q})$.

3.2 Various types of rigidity

This section introduces basic concepts about various types of rigidity and discusses the relation between these types.

3.2.1 Local rigidity

The most visual way to talk about local rigidity is to use continuous deformations as in Chapter 1. We formalize this definition as follows. A *continuous motion* from a framework $(G, \mathbf{p})$ to a framework $(G, \mathbf{q})$ in dimension d is a map

$$P : [0, 1] \times V \to \mathbb{R}^d$$

such that

1. $P(0, v) = p(v)$ for all $v \in V$,
2. $P(1, v) = q(v)$ for all $v \in V$,
3. $P(\cdot, v)$ is continuous for all $v \in V$,
4. $\|P(t, u) - P(t, v)\| = \|p(u) - p(v)\|$ for all $uv \in E(G)$ and $t \in [0, 1]$.

We say that P is a *smooth motion* if P is a continuous motion and $P(\cdot, v)$ is infinitely differentiable for each $v \in V$.

Gluck [43] showed that the following three definitions of local rigidity are equivalent.

Definition 3.2.1 (Local rigidity–continuous)**.** *A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is said to be locally rigid if every continuous motion of $(G, \mathbf{p})$ in $\mathbb{R}^d$ results in a framework $(G, \mathbf{q})$ that is congruent to $(G, \mathbf{p})$.*

Definition 3.2.2 (Local rigidity–topological)**.** *A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is locally rigid if there exists a neighborhood $N_p \subset \mathbb{R}^{nd}$ of $\mathbf{p}$ such that for every equivalent realization $(G, \mathbf{q})$ with $\mathbf{q} \in N_p$ we must have that $(G, \mathbf{q})$ is congruent to $(G, \mathbf{p})$.*

Definition 3.2.3 (Local rigidity–analytic)**.** *A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is locally rigid if every smooth motion of $(G, \mathbf{p})$ in $\mathbb{R}^d$ results in a framework $(G, \mathbf{q})$ that is congruent to $(G, \mathbf{p})$.*

The topological definition above gave rise to the term “local rigidity”.

Definition 3.2.4 (Flexible)**.** *A framework is called flexible in $\mathbb{R}^d$ if it is not locally rigid in $\mathbb{R}^d$.*

Using the rigidity map, we can also restate the topological definition of local rigidity as follows.

> A d-dimensional framework $(G, \mathbf{p})$ is locally rigid if for a small enough neighborhood N_p of $\mathbf{p}$, $f_G^{-1}(f_G(\mathbf{p})) \cap N_p = f_{K(V)}^{-1}(f_{K(V)}(\mathbf{p})) \cap N_p$.

3.2.2 Infinitesimal rigidity and rigidity matrix

The problem of determining the rigidity of a framework becomes more tractable if we linearize the problem. Suppose that P is a smooth motion of $(G, \mathbf{p})$. Then

$$\|P(t,u) - P(t,v)\|^2 = \|p(u) - p(v)\|^2 \quad \text{for all } uv \in E(G) \text{ and } t \in [0,1].$$

Differentiating this equation at $t = 0$ and setting $\mu(u) = P'(t,u)|_{t=0}$, $\mu(v) = P'(t,v)|_{t=0}$, we have

$$(p(u) - p(v))(\mu(u) - \mu(v)) = 0, \quad \text{for all } uv \in E(G).$$

Here, $\mu(u)$ can be regarded as the instantaneous velocity of the joint u at time $t = 0$. This motivates us to define the infinitesimal motions of a framework as follows.

Definition 3.2.5 (Infinitesimal motions)**.** *An infinitesimal motion of a d-dimensional framework $(G, \mathbf{p})$ is an assignment, to each vertex $v \in V$, of a vector $\mu(v) \in \mathbb{R}^d$ such that*

$$(p(u) - p(v))(\mu(u) - \mu(v)) = 0, \quad \textit{for all } uv \in E(G).$$

We can view an infinitesimal motion as a vector $\mu = (\dots, \mu(u), \dots)^T \in \mathbb{R}^{d|V|}$. Then it is not difficult to show that the infinitesimal motions of a framework $(G, \mathbf{p})$ form a vector subspace of $\mathbb{R}^{d|V|}$.

Among the smooth motions of a framework, there are those arising from the congruent motions of space. The infinitesimal motions induced by these motions are called *trivial infinitesimal motions*. Again, these trivial infinitesimal motions form a vector subspace of the infinitesimal motion subspace.

Definition 3.2.6 (Infinitesimal rigidity)**.** *A framework $(G, \mathbf{p})$ is infinitesimally rigid if its only infinitesimal motions are trivial ones. Otherwise, it is said to be infinitesimally flexible.*

When the affine dimension of $\{p(v) : v \in V\}$ is d, the dimension of the trivial infinitesimal motion space is $\binom{d+1}{2}$ (c.f. Section 2.6). Roughly speaking, there are

d independent infinitesimal motions induced by translations and $\binom{d}{2}$ independent infinitesimal motions induced by rotations. When this affine dimension is strictly less than d, i.e., when all the vertices lie in a hyperplane of $\mathbb{R}^d$, all infinitesimal motions are trivial if and only if G is a complete graph on at most d vertices [47].

Given a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$, the question whether $(G, \mathbf{p})$ is infinitesimally rigid can be answered by considering the rank of the so-called rigidity matrix.

Definition 3.2.7 (Rigidity matrix)**.** *The rigidity matrix $R(G, \mathbf{p})$ of a d-dimensional framework $(G, \mathbf{p})$ is a $|E| \times d|V|$ matrix whose rows are indexed by the edges of G and whose columns are indexed by the vertices of G such that*

- *each edge indexes one row;*
- *each vertex indexes d columns;*
- *the submatrix indexed by an edge $e = uv$ and the vertex u and v are $p(u) - p(v)$ and $p(v) - p(u)$ respectively;*
- *elsewhere all entries are zero.*

That is, $R(G, \mathbf{p})$ is written as

$$\begin{array}{cc} & \begin{array}{ccccc} & u & & v & \end{array} \\ e{=}uv & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots 0 \cdots & p(u) - p(v) & \cdots 0 \cdots & p(v) - p(u) & \cdots 0 \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

For example, the rigidity matrix of the framework in Figure 3.1 is

$$\begin{array}{cc} & \begin{array}{cccccccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} \end{array} \\ \begin{array}{c} \{1{,}2\} \\ \{1{,}3\} \\ \{1{,}4\} \\ \{2{,}3\} \\ \{3{,}4\} \end{array} & \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

It is easy to see that the rigidity matrix is in fact half of the Jacobian of the rigidity map, i.e,

$$R(G, \mathbf{p}) = \frac{1}{2} \mathrm{d} f_G|_{\mathbf{p}}.$$

It is also immediate from the definition that $\mu \in \mathbb{R}^{d|V|}$ is an infinitesimal motion of $(G, \mathbf{p})$ if and only if μ is in the nullspace of $R(G, \mathbf{p})$. Moreover, we have the following.

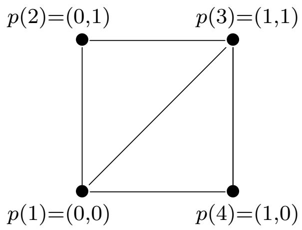


Figure 3.1: A planar framework.

Proposition 3.2.1 ([9]). *Let $(G, \mathbf{p})$ be a framework on n vertices in $\mathbb{R}^d$ with $|V| \geq 2$. Then the rank of $R(G, \mathbf{p})$ is at most $S(n, d)$ where*

$$S(n, d) = \begin{cases} dn - \binom{d+1}{2}, & n \geq d+1; \\ \binom{n}{2}, & n \leq d. \end{cases}$$

Moreover, a framework $(G, \mathbf{p})$ on n vertices in $\mathbb{R}^d$ is infinitesimally rigid if and only if $\operatorname{rank} R(G, \mathbf{p}) = S(n, d)$.

It is worth remarking that $S(n, d)$ is also the upper bound for the rank of the rigidity matrix of a complete framework on n vertices.

3.2.3 Local rigidity versus infinitesimal rigidity

The infinitesimal rigidity is in fact a stronger property than local rigidity.

Theorem 3.2.2. *If a framework $(G, \mathbf{p})$ is infinitesimally rigid then it is locally rigid.*

Several different proofs for this well-known fact can be found in [22] for example. Here we brief a proof idea by Alexandrov and Gluck [43]. If G has at most d vertices then by Proposition 3.2.1, $(G, \mathbf{p})$ is infinitesimally rigid if and only if G is a complete graph, so obviously $(G, \mathbf{p})$ is locally rigid. So suppose that G has at least $d+1$ vertices. The fact that $(G, \mathbf{p})$ is infinitesimally rigid implies that $\mathbf{p}$ is a regular point of f_G and of $f_{K(V)}$ as well. Therefore, for a small enough neighborhood N_p of $\mathbf{p}$, $f_G^{-1}(f_G(\mathbf{p})) \cap N_p$ is a manifold whose co-dimension is $\operatorname{rank} R(G, \mathbf{p})$. But $\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} R(K(V), \mathbf{p})$ is also the co-dimension of the manifold $f_{K(V)}^{-1}(f_{K(V)}(\mathbf{p})) \cap N_p$. This latter manifold is obviously a submanifold of the former one hence the equality between their co-dimension implies that they coincide. Therefore, $(G, \mathbf{p})$ is locally rigid.

The converse of Theorem 3.2.2 is not always true. Figure 3.2(a) illustrates a 2-dimensional framework which is locally rigid but not infinitesimally rigid. A slightly different embedding of the same graph as in Figure 3.2(b) is however both locally rigid and infinitesimally rigid. In fact, the first embedding is "special" in some sense: the three vertices b, e, c are collinear. It is not the only "special embedding" that may cause the difference between local rigidity and infinitesimal rigidity. Figure 3.2(c) shows a 2-dimensional framework without three collinear vertices which is locally rigid but not infinitesimally rigid. Considering only *generic embeddings* helps us avoiding this unconvenient situation.

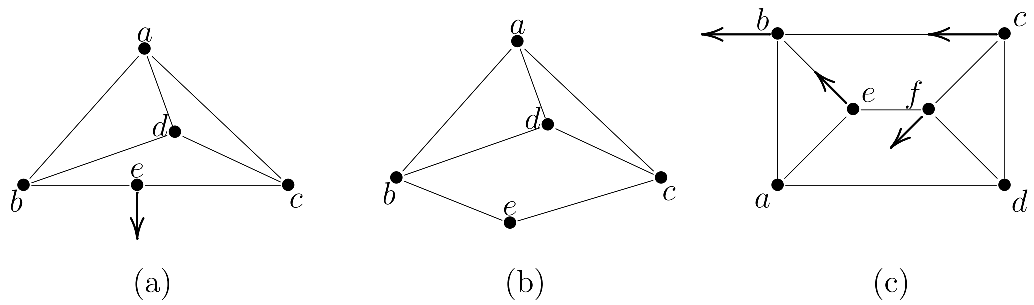


Figure 3.2: (a) There is a nontrivial infinitesimal motion with $\mu(a) = \mu(b) = \mu(c) = \mu(d) = 0$ and $\mu(e)$ perpendicular to the line *bec*.
(b) A generic embeding of the same graph as in (a). No non-trivial infinitesimal motion exists.
(c) A non-generic framework where *abcd* is a rectangle and *abe*, *bcf* are equilateral right triangles. A non-trivial infinitesimal motion is depicted.

Definition 3.2.8 (Linearly generic embedding)**.** *An embedding* $\mathbf{p} : V \to \mathbb{R}^d$ *is (linearly) generic if every submatrix of* $R(K(V), \mathbf{p})$ *attains the maximum rank over all d-dimensional embeddings. A framework* $(G, \mathbf{p})$ *with* $\mathbf{p}$ *being a generic embedding is called a generic framework or generic realization of* G*.*

Theorem 3.2.3 (Asimow and Roth [9])**.** *A generic framework is locally rigid if and only if it is infinitesimally rigid.*

Rigidity matroids

An important property of generic embeddings is that for two arbitrary generic embeddings and a subset F of edges, the rank of the set of rows indexed by F in the two rigidity matrices are the same. Therefore, all generic embeddings define a unique linear matroid on the edge set E of G. We call this matroid the d-dimensional *generic rigidity matroid* of G. This matroid can be regarded as the restriction on $E(G)$ of the d-dimensional generic rigidity matroid of the complete graph $K(V)$. We denote the latter by $\mathcal{G}_d(n)$ where $n = |V|$. Note that in this matroid, edge sets that induce isomorphic subgraphs have the same rank, namely, this matroid depends uniquely on n and d. Again, if $n \leq n'$, $\mathcal{G}_d(n)$ can be considered as a restriction of $\mathcal{G}_d(n')$, so sometimes we will use $\mathcal{G}_d$ to mean a large generic rigidity matroid that contains all the d-dimensional generic rigidity matroids considered in our context.

3.2.4 Static rigidity

Let us consider rigidity from a structural engineering viewpoint. Suppose that at each joint v of a 3-dimensional bar-joint structure we apply a force $\mathbf{f}(v)$ such that the net force and the net moment (about three coordinate axes) on the whole structure is zero. Such a system $\mathbf{F} = (\dots, \mathbf{f}(v), \dots)$ of forces is called an *equilibrium force*. The equilibrium condition is equivalent to

$$\sum_{v \in V} \mathbf{f}(v) = \mathbf{0} \quad \text{and} \quad \sum_{v \in V} \mathbf{f}(v) \times p(v) = \mathbf{0}, \tag{3.2.1}$$

which is a system of 6 linearly independent equations. The first three equations are for the three components of the net force and the last three equations are for the net moment about three coordinate axes. It follows that the space of all equilibrium forces of a 3-dimensional structure is a vector space of dimension $3|V| - 6$. The structure is stable if for every equilibrium force, it can avoid the deformation by generating tensions and compressions on its bars – we say that the equilibrium force is *resolved*.

The system (3.2.1) can be rewritten using join as

$$\sum_{v \in V} (\mathbf{f}(v), 0) \vee (p(v), 1) = \mathbf{0}.$$

where $(\mathbf{f}(v), 0)$ and $(p(v), 1)$ are 4-dimensional vectors obtained from $\mathbf{f}(v), p(v)$ by adding a 0 and a 1 entry respectively. For bar-joint frameworks in general dimensions d, we say that a system of forces $\mathbf{F} = (\dots, \mathbf{f}(v), \dots)$, $\mathbf{f}(v) \in \mathbb{R}^d$, exerted on the joints, is in equilibrium if

$$\sum_{v \in V} (\mathbf{f}(v), 0) \vee (p(v), 1) = \mathbf{0}.$$

(See [27] for more details.)

Therefore, all equilibrium forces of a d-dimensional bar-joint framework of full dimension form a vector space which has dimension $d|V| - \binom{d+1}{2}$.

Definition 3.2.9 (Static rigidity)**.** *A framework $(G, \mathbf{p})$ is statically rigid if for every equilibrium force $\mathbf{F} = (\dots, \mathbf{f}(u), \dots)$ there is an assignment of scalar ω_{uv} to each edge $uv \in E$ such that*

$$\mathbf{f}(u) + \sum_{uv \in E} \omega_{uv}(p(v) - p(u)) = \mathbf{0}, \quad \textit{for every } u \in V. \tag{3.2.2}$$

Here, ω_{uv} plays the role of the internal stress on the bar uv, which generates an internal force $\omega_{uv}(p(u) - p(v))$ at the joint u. Equation (3.2.2) means that the sum of the external force and the internal forces at each joint is zero.

3.2.5 Infinitesimal rigidity versus static rigidity

Let $\mathbf{F} = (\dots, \mathbf{f}(u), \dots)$ be a system of forces in equilibrium on a framework $(G, \mathbf{p})$ which is resolved by a stress $\omega = (\dots, \omega_{uv}, \dots)$. Using the rigidity matrix, the system of linear equations (3.2.2) can be rewritten as

$$\mathbf{F} - \omega R(G, \mathbf{p}) = \mathbf{0},$$

which means that $\mathbf{F}$ belongs to the row space of $R(G, \mathbf{p})$. Conversely, if $\mathbf{F} = aR(G, \mathbf{p})$, where $a \in \mathbb{R}^{|E(G)|}$ is a vector in the row space of $R(G, \mathbf{p})$ then it is an equilibrium force and obviously can be resolved with $\omega = a$. Therefore, the framework $(G, \mathbf{p})$ is statically rigid if and only if the row space of $R(G, \mathbf{p})$ has dimension $d|V| - \binom{d+1}{2}$. But it is also the necessary and sufficient condition for $(G, \mathbf{p})$ to be infinitesimally rigid. Hence we obtain the equivalent between the static rigidity and the infinitesimal rigidity of bar-joint frameworks.

Theorem 3.2.4 (see, e.g., [47, 22])**.** *A framework $(G, \mathbf{p})$ is infinitesimally rigid if and only if it is statically rigid.*

3.2.6 Global rigidity

Definition 3.2.10 (Global rigidity)**.** *A framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is globally rigid if for every framework $(G, \mathbf{q})$ in $\mathbb{R}^d$, $(G, \mathbf{q})$ is equivalent to $(G, \mathbf{p})$ implies that $(G, \mathbf{q})$ is congruent to $(G, \mathbf{p})$.*

From this definition, a globally rigid framework is clearly locally rigid.

3.2.7 Universal rigidity

So far, we have always considered frameworks and their motions in the same dimension. As discussed in Chapter 1, the uniqueness of the location for a localization problem, which is equivalent to the global rigidity of the associated framework, does not imply that we can find this location efficiently. The universal rigidity of a framework is a stronger property that guarantees the efficiency of the exploited SDP method.

Definition 3.2.11 (Universal rigidity)**.** *A framework $(G, \mathbf{p})$ is said to be universally rigid if and only if, for every framework $(G, \mathbf{q})$ in* any *dimension, $(G, \mathbf{q})$ is equivalent to $(G, \mathbf{p})$ implies that $(G, \mathbf{q})$ is congruent to $(G, \mathbf{p})$.*

Universal rigidity as universal local rigidity

It is clear from the definitions that universal rigidity is a stronger property than global rigidity which is again stronger than local rigidity. On the other hand, the following result of Bezdek and Connelly shows that, if $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are two realizations of G in dimension $d \le d'$ respectively, then there is a smooth motion from $(G, \mathbf{p})$ to $(G, \mathbf{q})$ in dimension $2d'$. (We regard $(G, \mathbf{p})$ and $(G, \mathbf{q})$ as embedded in dimension $2d'$ by adding zeros to missing coordinates.)

Lemma 3.2.5 (Leapfrog Lemma [14])**.** *Suppose that $\mathbf{p}$ and $\mathbf{q}$ are two embeddings of G in $\mathbb{R}^m$. Then the following $P(t, v)$ is a smooth motion in $\mathbb{R}^{2m}$ such that $P(0, v) = p(v), P(1, v) = q(v)$ for $v \in V$ and for $0 \le t \le 1$, $\|P(t, u) - P(t, v)\|$ is monotone for every $u, v \in V$:*

$$P(t, v) = \Big(\frac{p(v) + q(v)}{2} + (\cos \pi t)\frac{p(v) - q(v)}{2}, (\sin \pi t)\frac{p(v) - q(v)}{2}\Big), \quad \textit{for } v \in V.$$

This result implies that universal rigidity can be regarded as local rigidity in a universal sense.

3.2.8 Stress matrices

Definition 3.2.12 (Self-stress)**.** *A self-stress or an equilibrium stress of a framework $(G, \mathbf{p})$ is an assignment of a scalar $\omega_{uv} = \omega_{vu}$ to each edge $uv \in E$ such that for each vertex $u \in V$ the following* equilibrium condition *is satisfied.*

$$\sum_{v \in V : uv \in E} \omega_{uv}(p(u) - p(v)) = \mathbf{0}. \qquad (3.2.3)$$

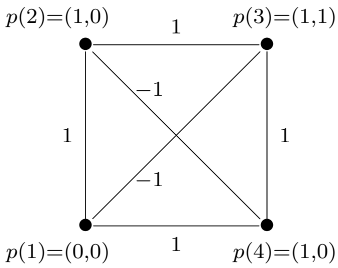


Figure 3.3: A framework with an equilibrium stress.

From the statics viewpoint discussed above, a self-stress is a stress of the bar-joint structure which resolves the zero equilibrium force. It is also immediate from the definition that $\omega = (\omega_{uv} : uv \in E)$ is an equilibrium stress of $(G, \mathbf{p})$ if and only if $\omega R(G, \mathbf{p}) = \mathbf{0}$.

Definition 3.2.13 (Stress matrix)**.** *Let $(G, \mathbf{p})$ be a framework and $\omega = (\omega_{uv} : uv \in E(G))$ be an equilibrium stress of $(G, \mathbf{p})$. The stress matrix of $(G, \mathbf{p})$ associated with ω is the $|V| \times |V|$ symmetric matrix Ω with rows and columns indexed by vertices in V such that*

$$\Omega_{uv} = \begin{cases} -\omega_{uv} & \textit{if } uv \in E, \\ \sum_{w \in V : uw \in E} \omega_{uw} & \textit{if } u = v, \\ 0 & \textit{otherwise.} \end{cases}$$

For example, the stress matrix associated with the stress in Figure 3.3 is

$$\Omega = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \end{array}.$$

Global rigidity and universal rigidity via stress matrices

It turns out that stress matrices encode rigidity properties of frameworks. Below we summarize some well-known results on the relations between stress matrices and rigidity of frameworks. These results are explained with more details in Chapter 8. First we need one more definition.

Definition 3.2.14 (Algebraically generic embedding)**.** *An embedding $\mathbf{p} : V \to \mathbb{R}^d$ is algebraically generic if the set of all the coordinates of $\mathbf{p}$ is algebraically independent over the rationals.*

Theorem 3.2.6 (Connelly [23], Gortler, Healy and Thurston [44])**.** *An algebraically generic framework* $(G, \mathbf{p})$ *on* $n \geq d+2$ *vertices in* $\mathbb{R}^d$ *is globally rigid if and only if* $(G, \mathbf{p})$ *possesses a stress matrix of rank* $n-d-1$*.*

Theorem 3.2.7 (Connelly [21, 19], Alfakih [3], Gortler and Thurston [45])**.** *An algebraically generic framework* $(G, \mathbf{p})$ *on* $n \geq d+2$ *vertices in* $\mathbb{R}^d$ *is universally rigid if and only if* $(G, \mathbf{p})$ *possesses a positive semidefinite stress matrix of rank* $n-d-1$*.*

Furthermore, Alfakih and Ye [8] shows that the "if" part of Theorem 3.2.7 still holds if one relaxes the algebraic genericity condition on the embedding to a general position condition, which requires that, under the map $\mathbf{p}$, every $d+1$ vertices of V are mapped to affinely independent points in $\mathbb{R}^d$. This result will be strengthened in Chapter 8.

For the sake of convenience, in passages concerning different types of rigidity, we will use the term "generic" both for "linearly generic" if it is about local/infinitesimal rigidity and for "algebraically generic" if it is about global/universal rigidity.

3.3 Combinatorial rigidity

One central problem of rigidity theory is that given a framework, decide whether it is rigid (locally, infinitesimally, globally or universally) or not. This problem for an arbitrary embedding is known to be untractable, in general [91], [1]. The equivalence between infinitesimal rigidity and local rigidity for generic frameworks as well as the characterization of globally and universally rigid algebraically generic frameworks through stress matrices suggest that this problem is more tractable for generic frameworks. On the other hand, it often turns out that if a rigidity property holds for a generic realization $(G, \mathbf{p})$ of a graph G then it also holds for every generic realization $(G, \mathbf{q})$ of G in the same dimension. If this happens, the rigidity property in question is said to be a *generic property*. More generally, we define:

Definition 3.3.1 (Generic property)**.** *A property* P *of frameworks is* generic *in dimension* d *if whenever some generic realization* $\mathbf{p}$ *of a graph* G *in* $\mathbb{R}^d$ *has property* P*, then every generic realization* $\mathbf{q}$ *of* G *in* $\mathbb{R}^d$ *also has property* P*.*

The above discussion about local rigidity and infinitesimal rigidity tells us that local rigidity and infinitesimal rigidity are both generic properties of bar-joint frameworks in every dimension. More precisely, given a graph G, if there exists a d-dimensional linearly generic realization $(G, \mathbf{p})$ of G that is locally (resp., infinitesimally) rigid then every d-dimensional generic realization $(G, \mathbf{q})$ of G is also locally (resp., infinitesimally) rigid since the rank of $R(G, \mathbf{p})$ and $R(G, \mathbf{q})$ are equal. Global rigidity is also proved to be a generic property by Gortler, Healy and Thurston [44]. However, universal rigidity is not a generic property (Figure 1.5).

When focusing on graphs, we say that a graph G has property P in dimension d if every generic framework of G in dimension d has property P. Note that then locally rigid, infinitesimally rigid and statically rigid mean the same thing for graphs.

The study of combinatorial rigidity focuses on finding combinatorial characterizations of the underlying graphs of generic frameworks that have some rigidity property. The first fundamental result in combinatorial rigidity is the famous theorem of Laman which characterizes the underlying graphs of locally rigid generic frameworks in the plane. We refer to these graphs as *generically locally rigid graphs* in $\mathbb{R}^2$.

Theorem 3.3.1 (Laman [75]). *A graph $G = (V, E)$ is generically locally rigid in $\mathbb{R}^2$ if and only if E contains a subset F that satisfies*

1. $|F| = 2|V| - 3$, *and*
2. $|F'| \leq 2|V(F')| - 3$ *for every subset* $F' \subseteq F$.

In other words, a graph is generically locally rigid in $\mathbb{R}^2$ if and only if it contains a $(2, 3)$-tight spanning subgraph.

In many cases, it is convenient to consider a minimally generically locally rigid graph, i.e., a generically locally rigid graph with the property that deleting any edge makes it no more generically locally rigid. These graphs are often referred to as *isostatic graphs*. The following results are equivalent forms of Laman's theorem for 2-dimensional isostatic graphs.

Theorem 3.3.2 (Lovász and Yemini [80], Recski [89]). *For a graph G, the following statements are equivalent.*

1. *G is isostatic in the plane.*
2. *Duplicating any edge of G results in a graph that is the union of two edge-disjoint spanning trees.*

3. *Adding an edge between any two vertices of G results in a graph that is the union of two edge-disjoint spanning trees.*

Theorem 3.3.3 (Crapo [26]). *A graph $G = (V, E)$ is isostatic in the plane if and only if G can be decomposed into three edge-disjoint trees T_1, T_2, T_3 such that each vertex in V is covered by exactly two of them, and no two subtrees of T_1, T_2, T_3 with more than one vertex span the same set of vertices.*

A decomposition satisfying the condition in this theorem is call a $3T2$ *proper decomposition.*

From a matroid viewpoint, the generic local rigidity of a graph in the plane can be determined from the rank function of the generic rigidity matroid $\mathcal{G}_2$.

Theorem 3.3.4 (Lovász and Yemini [80]). *The rank of $\mathcal{G}_2$ is given by*

$$r_{\mathcal{G}_2}(E) = \min\{\sum_{i=1,\ldots,m} (2|V_i| - 3) : E(V_1), \ldots E(V_m) \text{ partition } E\}.$$

One important trend in rigidity theory is to study inductive construction of generically rigid graphs. Complete characterizations of generically locally rigid graphs and generically globally rigid graphs in terms of inductive construction are obtained for bar-joint model in dimension 2. (They are trivial for dimension 1.) This approach is even more fruitful in other models such as body-bar frameworks, body-bar-hinge frameworks, etc., where complete characterizations for generic local rigidity are obtained for all dimensions. Below we summarize results on generic local and global rigidity related to inductive construction.

First, we recall the definition of some useful extension operations.

Definition 3.3.2 (0-extension, 1-extension).

1. *A d-dimensional 0-extension on a graph H is an operation that adds to H a new vertex v and connects v to d different vertices $v_1, \ldots, v_d$ in $V(H)$ (Figure 3.4).*

2. *A d-dimensional 1-extension on a graph H deletes an existing edge v_1v_2 in H, adds to H a new vertex v, then connects v to v_1, v_2 and other $d-1$ vertices $v_3, \ldots, v_{d+1}$ in H (Figure 3.5).*

A graph G obtained from a graph H by a d-dimensional 0-extension (resp., 1-extension) is called a d-dimensional 0-extension (resp., 1-extension) of H. 0-extension and 1-extension are known to preserve the isostaticity of a graph.

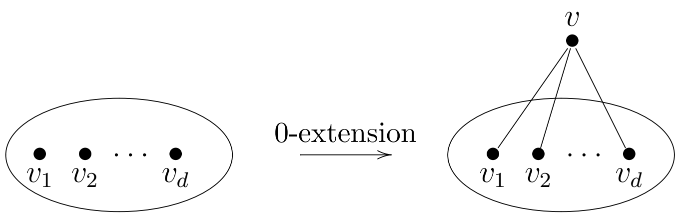


Figure 3.4: d-dimensional 0-extension.

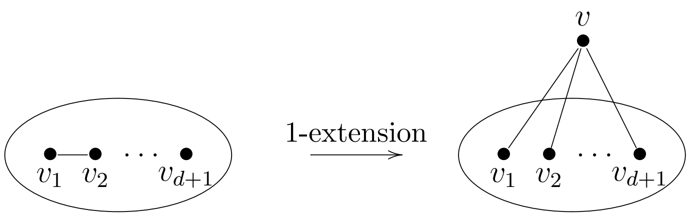


Figure 3.5: d-dimensional 1-extension.

Theorem 3.3.5 (see [10, 102])**.** *Suppose that H is a d-dimensional isostatic graph and G is obtained from H by a d-dimensional 0-extension or 1-extension. Then G is d-dimensional isostatic.*

In dimension 2, these two operations are proved to be sufficient for characterizing isostatic graphs.

Theorem 3.3.6 (see, e.g., [10, 47, 58])**.** *A graph $G = (V, E)$ is isostatic in the plane if and only if G can be constructed from K_2 by a sequence of 2-dimensional 0-extensions and 1-extensions. (See Figure 3.6.)*

The problem of characterizing isostatic graphs in dimension 3 seems to be extremely difficult. An example showing that the Laman-type counting condition does not work in dimension 3 is the famous *double banana* in Figure 3.7. It is easy to verify that, in this graph, for every subset X of V with $|X| \geq 3$, $i(X) \leq 3|X| - 6$ and $|E| = 3|V| - 6$. Yet the graph is not rigid in dimension 3 since there is a relative rotation of the two bananas about the axis through their two ends.

A characterization of 3-dimensional isostatic graphs in terms of inductive construction may have to consider the following conjecture.

Conjecture 3.3.7 (X-replacement [102])**.** *Let G be a 3-dimensional isostatic graph and $v_1, \ldots, v_5$ distinct vertices of G with $v_1v_2, v_3v_4 \in E(G)$. Then the operation that deletes v_1v_2, v_3v_4 and adds to G a new vertex v and edges $vv_1 \ldots, vv_5$ results in a 3-dimensional isostatic graph.*

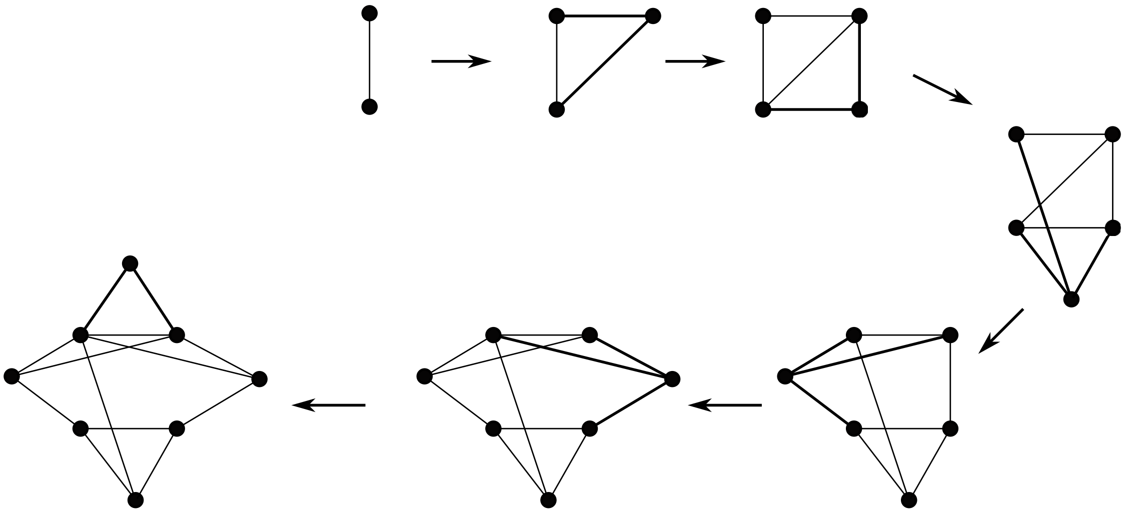


Figure 3.6: A sequence of graphs constructed from K_2 by a sequence of 2-dimensional 0-extensions and 1-extensions. Bold edges denote new edges. All these graphs are isostatic.

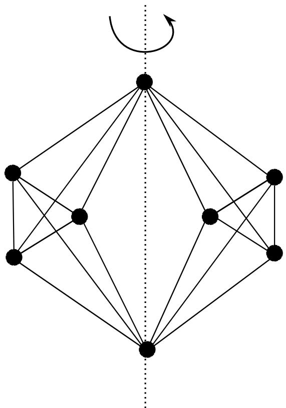


Figure 3.7: The double banana.

Although a combinatorial characterization of 3-dimensional generically locally (globally) rigid graphs remains challenging, a positive result is obtained for an important class of graphs: square graphs (equivalently called *molecular graphs* for their role in modeling molecular structures). The *square graph* G^2 of a graph G is obtained from G by adding an edge between every pair of vertices that have a common neighbor in G (c.f. Figure 1.3). The Molecular Conjecture, proved by Katoh and Tanigawa [72] for body-hinge structures in general dimension, implies the following combinatorial characterization for the 3-dimensional rigidity of square graphs (see [59]).

Theorem 3.3.8 (Katoh and Tanigawa [72]). *The square graph G^2 of a graph G is* *generically locally rigid in dimension* 3 *if and only if the multigraph* $5G$*, obtained from* G *by multiplying each edge of* G *by* 5*, contains* 6 *edge-disjoint spanning trees.*

In dimension one, it is a well known fact that a graph is generically globally rigid if and only if it is 2-connected. The problem of combinatorially characterizing 2-dimensional generically globally rigid graphs was settled by Jackson and Jordán. A graph is *redundantly generically locally rigid* in dimension d if deleting any edge results in a generically locally rigid graph in dimension d.

Theorem 3.3.9 (Hendrickson [53], Jackson and Jordán [58])**.** *A graph is generically globally rigid in dimension* 2 *if and only if it is* 3*-connected and redundantly generically locally rigid in dimension* 2*.*

In fact, the "only if" part of this theorem is shown by Hendrickson [53] for all dimensions. He conjectured the truth of the converse for all dimensions. Jackson and Jordán confirm it for dimension 2 by proving an inductive construction of graphs that are 3-connected and redundantly locally rigid in the plane: these graphs are constructed from K_4 by a sequence of edge-additions and 2-dimensional 1-extensions, which are known to preserve the generic global rigidity. For $d = 3$, Connelly proved that $K_{5,5}$ is a counter example. Though infinite families of counter examples for Hendrickson's conjecture in higher dimensions ($d \geq 5$) are obtained [40], it remains an open question whether $K_{5,5}$ is the only counter example in $\mathbb{R}^3$.

Unlike generic local rigidity and global rigidity, knowledge on combinatorial properties of generically universally rigid graphs is quite modest even in dimension one. Probably, the only known construction to create generically universally rigid graphs is the following.

Lemma 3.3.10 (Ratmanski [88])**.** *A graph* G *on at least* $d + 2$ *vertices is* d*-dimensional generically universally rigid (d-GUR) if* G *can be obtained from* K_{d+1} *by the following operations:*

(i) *add an edge,*

(ii) *choose two graph* G_1, G_2 *built by these operations, choose two sets* U_1, U_2 *of each with* $|U_1| = |U_2| \geq d + 1$*, delete all edges joining vertices of* U_1 *in* G_1*, then glue the two graphs together along the vertices in* U_1 *and* U_2*.*

In particular, if we add a vertex to a d-GUR graph G and connect it to at least $d+1$ vertices of G then we obtain a d-GUR graph. It is an open question whether the converse of Lemma 3.3.10 is true.

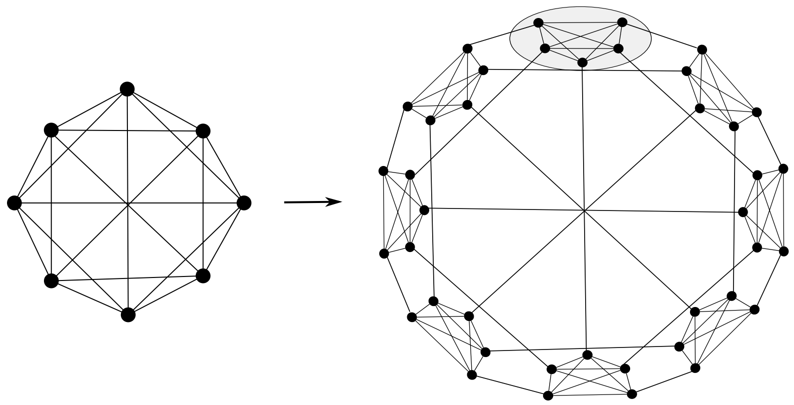


Figure 3.8: A 5-connected graph that is not generically rigid in $\mathbb{R}^2$ [80].

An interesting question in combinatorial rigidity theory is that whether high vertex-connectivity implies generic local/global rigidity. For dimension 2, based on Laman's characterization, Lovász and Yemini [80] showed that every 6-vertex-connected graphs minus any three edges are generically locally rigid in $\mathbb{R}^2$. They also pointed out that 6 is the minimum value, i.e., there are 5-vertex-connected graphs that are not generically locally rigid in $\mathbb{R}^2$ (Figure 3.8). Combining the result of Lovász and Yemini with Theorem 3.3.9, it yields that 6-vertex-connected graphs are generically globally rigid in $\mathbb{R}^2$.

In dimension $d \geq 3$, as for characterization of generic local rigidity, the relation between vertex-connectivity and generic local/global rigidity of graphs is unknown. It is conjectured by Lovász and Yemini that $d(d+1)$-connectivity is sufficient. For molecular graphs, using the known combinatorial characterization proved by Katoh and Tanigawa (Theorem 3.3.8), Jordán [69] derived that every 7-vertex-connected molecular graphs are generically locally rigid in $\mathbb{R}^3$.

The same question about the relation between vertex-connectivity and the generical universal rigidity of graphs would be asked. However, in Chapter 8 we prove that no vertex-connectivity can guarantee the generic universal rigidity of graphs in any dimension. In fact, we show that every complete bipartite graph, except K_2, is non generically universally rigid in any dimension.

Chapter 4

Matroid approach

Contents

4.1 Introduction

The concept of abstract rigidity matroid was first introduced by Graver [46] as a generalization of the generic rigidity matroid. Yet application of abstract rigidity matroids in the study of rigidity is still limited, viewing the generic rigidity matroid as an abstract rigidity matroid allows one to concentrate on its combinatorial nature. On the other hand, abstract rigidity matroids are an interesting topic in its own right with many open questions and may have applications in other problems as discussed in Section 4.5.

Before going into the definition of abstract rigidity matroids let us keep in mind the two simple but important properties of d-dimensional generic bar-joint frameworks whose detailed proof can be found in [47]. The first one is that, if two frameworks are glued together over at most $d-1$ joints, then the composed framework is not rigid (it always allows a relative rotation between the two parts about a $d-2$-dimensional affine space containing the glued joints). Moreover, if we add any bar crossing these two parts then the framework becomes "more rigid". The second one is that, if two rigid frameworks are glued together over at least d joints, then the obtained framework is also rigid. We can formalize these observations as follows. In this chapter we abuse notation $K(V)$ to denote also the edge set of the complete graph on a vertex set V and K_t to denote the edge set of a complete graph on t vertices. Let $\mathcal{G}_d(n)$ be the generic rigidity matroid on the complete graph $K = (V, K(V))$ and r its rank function. An edge set $E \subseteq K(V)$ is said to be *rigid* if $\mathrm{cl}(E) = K(V(E))$, i.e., E spans $K(V(E))$. In matroid language, the two properties above are restated as follows.

(C1) If $|V(E) \cap V(F)| \leq d-1$, then $\mathrm{cl}(E \cup F) \subseteq K(V(E)) \cup K(V(F))$.

(C2) For every pair of rigid subsets E, F of $K(V)$, if $|V(E) \cap V(F)| \geq d$, then $E \cup F$ is rigid.

Let $\mathcal{A}_d$ be a matroid on $K(V)$ with closure operator $\mathrm{cl}_{\mathcal{A}_d}(\cdot)$. A subset E of $K(V)$ is said to be *rigid* (in $\mathcal{A}_d$) if $\mathrm{cl}_{\mathcal{A}_d}(E) = K(V(E))$. Then $\mathcal{A}_d$ is called a *d-dimensional abstract rigidity matroid* if it satisfies (C1) and (C2).

This chapter presents an approach to the problem of characterizing the generic rigidity matroid from a pure matroid viewpoint. First, in Section 4.2, we describe our result on combinatorial characterization of abstract rigidity matroids. Then, in Section 4.3, we introduce the concept of 1-extendable abstract rigidity matroid – a generalization that captures more properties of the generic rigidity matroid than abstract rigidity matroids – and show that although in dimension 2 a 1-extendable abstract rigidity matroid coincides with the generic rigidity matroid, in dimension 3 they can be different. Section 4.4 is devoted to the study of intersecting submodular functions that induce abstract rigidity matroids. We provide a necessary condition for these functions. With an additional assumption on the symmetry, we show that this necessary condition is also sufficient. We close the chapter with a discussion on a potential application of our results on abstract rigidity matroids. This work originates from the author's master's thesis and partly published in [85] before the enrolment to the PhD program. We include these parts in this thesis to provide a complete view of the early approach. So many details and proofs in these parts will be omitted.

4.2 Characterizing abstract rigidity matroids

A *vertex star* of the complete graph $(V, K(V))$ is the set of all edges incident to some vertex $v \in V$. In [47], Graver, Servatius and Servatius posed two questions on the characterization of abstract rigidity matroids in dimension 2.

> **Question 1** [47, page 107] Is it true that a matroid $\mathcal{M}$ on the edge set of the complete graph $(V, K(V))$ is a 2-dimensional abstract rigidity matroid if and only if all of the K_4's are circuits and all of the vertex stars minus an edge are cocircuits?

> **Question 2** [47, page 108] Is it true that a matroid $\mathcal{M}$ on the edge set of the complete graph $(V, K(V))$ is a 2-dimensional abstract rigidity matroid if and only if all of the K_4's are circuits and $r(K(U)) = 2|U|-3$ for all $U \subseteq V$ with $|U| \geq 2$?

Subsequently, in [48], they gave an affirmative answer to Question 1 together with its generalization in higher dimension. In [85], we show that the condition in Question 1 and the one in Question 2 are both equivalent to the property that $\mathcal{M}$ is an abstract rigidity matroid. This gives an affirmative answer to Question 2 and its generalization as well as an alternative proof to the result of Graver, Servatius and Servatius [48, Theorem 0.2]. As a byproduct, we obtain a polynomial algorithm for testing if a matroid given by an independence oracle is a d-dimensional abstract rigidity matroid for any fixed d.

For $E \subseteq K$, $v \in V \setminus V(E)$ and $u_1, \ldots, u_k \in V(E)$, we call $F = E + vu_1 + \cdots + vu_k$ a *k-valent* 0-*extension* of E. Let K_t denote the edge set of a complete subgraph on t vertices of (V, K). The principal ingredient to prove our characterization of abstract rigidity matroids is the following lemma.

Lemma 4.2.1 ([85])**.** *A matroid on the edge set of the complete graph (V, K) is a d-dimensional abstract rigidity matroid if and only if it satisfies:*

1. $r_{\mathcal{A}_d}(K(V)) = d|V| - d(d+1)/2$*;*
2. *Each k-valent 0-extension of an independent set of $\mathcal{A}_d$ is also an independent set of $\mathcal{A}_d$ for every $k \leq d$.*

The following theorem answers the two questions above and provides characterizations of d-dimensional abstract rigidity matroids for any $d \geq 2$.

Theorem 4.2.2 ([85])**.** *The following statements are equivalent for a matroid $\mathcal{A}_d$ on $K(V)$.*

(i) *$\mathcal{A}_d$ is a d-dimensional abstract rigidity matroid on K.*

(ii) *All K_{d+2}'s in $K(V)$ are circuits of $\mathcal{A}_d$ and all vertex stars minus $(d-1)$ edges are cocircuits of $\mathcal{A}_d$.*

(iii) *All K_{d+2}'s in $K(V)$ are circuits of $\mathcal{A}_d$ and $r_{\mathcal{A}_d}(K(U)) = d|U| - d(d+1)/2$ for every $U \subseteq V$ with $|U| \geq d+1$.*

(iv) *All K_{d+2}'s in $K(V)$ are circuits of $\mathcal{A}_d$ and $r_{\mathcal{A}_d}(K) = d|V| - d(d+1)/2$.*

Theorem 4.2.2 implies that we can discern whether a matroid $\mathcal{A}_d$ given by an independence oracle is a d-dimensional abstract rigidity matroid for a fixed positive integer d in polynomial time by checking condition (iv). Although it is not mentioned in [48], condition (ii) also implies a polynomial time algorithm for testing whether a given matroid is a d-dimensional abstract rigidity matroid. However, to check condition (ii), we would need to verify whether every vertex star minus $(d-1)$ edges is a cocircuit, or, equivalently, its complement is a hyperplane, which would take $O(n^{d+1})$ calls to the independence oracle, while an algorithm using condition (iv) would need only $O(n^d)$ oracle calls.

4.3 1-extendable abstract rigidity matroids

In this section, extensions on a graph are regarded as extensions on its edge set. A matroid $\mathcal{M}$ on the edge set K of the complete graph (V, K) is called a d-dimensional 1-*extendable abstract rigidity matroid* if $\mathcal{M}$ is a d-dimensional abstract rigidity

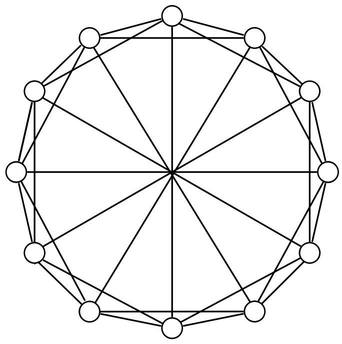


Figure 4.1: The graph H in the proof of Theorem 4.3.1. Each vertex is connected to four nearest vertices and its opposite vertex. $|H| = 30 = r(K_{12})$ in $\mathcal{G}_3(12)$.

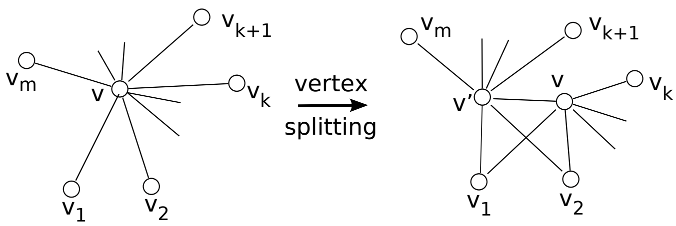


Figure 4.2: Vertex splitting operation

matroid on K and if $F \subseteq K$ is a d-dimensional 1-extension of an independent set in $\mathcal{M}$ then F is independent in $\mathcal{M}$.

The generic rigidity matroid on K is an example of a 1-extendable abstract rigidity matroid on K. In dimension 2, the generic rigidity matroid is characterized by Laman's condition and also by Theorem 3.3.6. A corollary of these characterizations is that the 2-dimensional generic rigidity matroid is the unique 2-dimensional 1-extendable abstract rigidity matroid (Graver, Servatius and Servatius [47, Theorem 4.2.3]). The following theorem shows that $\mathcal{G}_3$ is not the only 3-dimensional 1-extendable abstract rigidity matroid.

Theorem 4.3.1. *There exists a* 3*-dimensional* 1*-extendable abstract rigidity matroid that is not a generic rigidity matroid.*

We briefly describe the idea of the proof of Theorem 4.3.1. Let us consider the subset H of $K = K_{12}$ depicted in Figure 4.1.

For a subset $E \subset K$ with $vv_1, vv_2, \ldots, vv_m \in E$ and a vertex $v' \in V \setminus V(E)$ the edge set $F = E + vv' + v'v_1 + v'v_2 - vv_{k+1} - \cdots - vv_m + v'v_{k+1} + \cdots v'v_m$ with $2 \leq k \leq m$ is said to be obtained from E by a *vertex splitting* operation (Figure 4.2). In $\mathcal{G}_3$, if E is independent, then F is also independent (Whiteley [108]).

The graph H can be obtained from K_4 by a vertex splitting operation and a

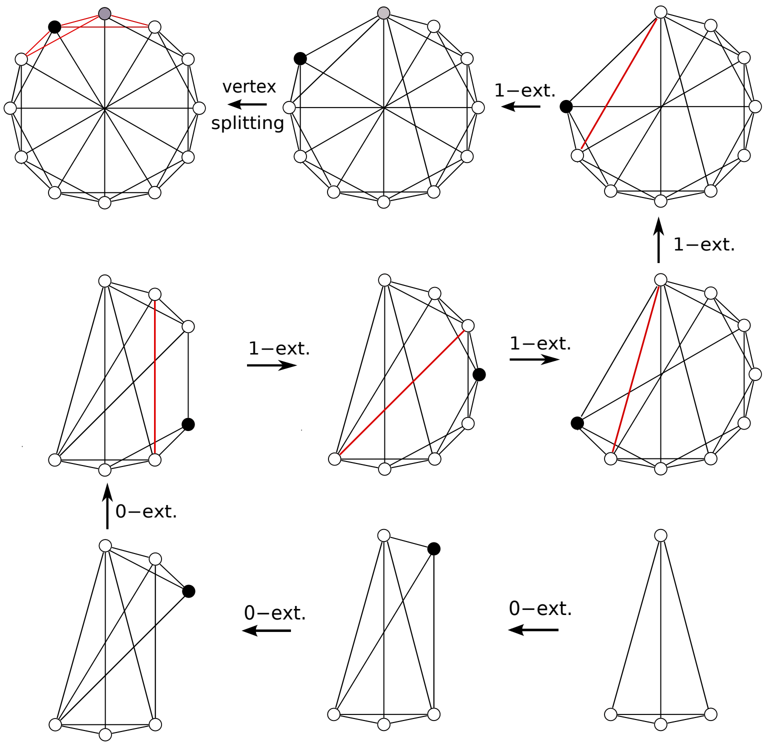


Figure 4.3: Building H from K_4 by a sequence of 0-extensions, 1-extensions and a vertex splitting operation. New vertices are denoted in black.

sequence of 1-extensions and 0-extensions as shown in Figure 4.3. Since K_4 is independent in the 3-dimensional generic rigidity matroid, $E(H)$ is also independent in $\mathcal{G}_3(12)$.

Furthermore, we can show that the subset $H - e + f$ is a base of K in $\mathcal{G}_3(12)$ for every $e \in H$ and $f \in K \setminus H$. Figure 4.4 illustrates how a graph $H + e - f$ can be constructed from K_4 by 0-extensions and 1-extensions.

Now, let $\mathcal{B}$ be the set of bases of $\mathcal{G}_3(12)$, then $H \in \mathcal{B}$. Let $\mathcal{B}' = \mathcal{B} - H$. Then using the fact that $E(H) - e + f$ is a base of K in $\mathcal{G}_3(12)$ for every $e \in H$ and $f \in K \setminus E(H)$ we can show that $\mathcal{B}'$ is a set of bases of a matroid $\mathcal{M}'$ on K. This matroid $\mathcal{M}'$ is obviously a 1-extendable abstract rigidity matroid since the only base that we delete from $\mathcal{G}_3(12)$ to obtain $\mathcal{M}'$ is the edge set of a graph with all vertices of degree 5.

Remark: Walter Whiteley communicated that, in dimension $d \geq 4$, it had been already known that the spline matroid is a 1-extendable abstract rigidity matroid which is distinct from the generic rigidity matroid. In dimension 3, however, the spline matroid is conjectured to be isomorphic to the rigidity matroids [110]. Moreover, the example in the proof of Theorem 4.3.1 is also showing that the inde-

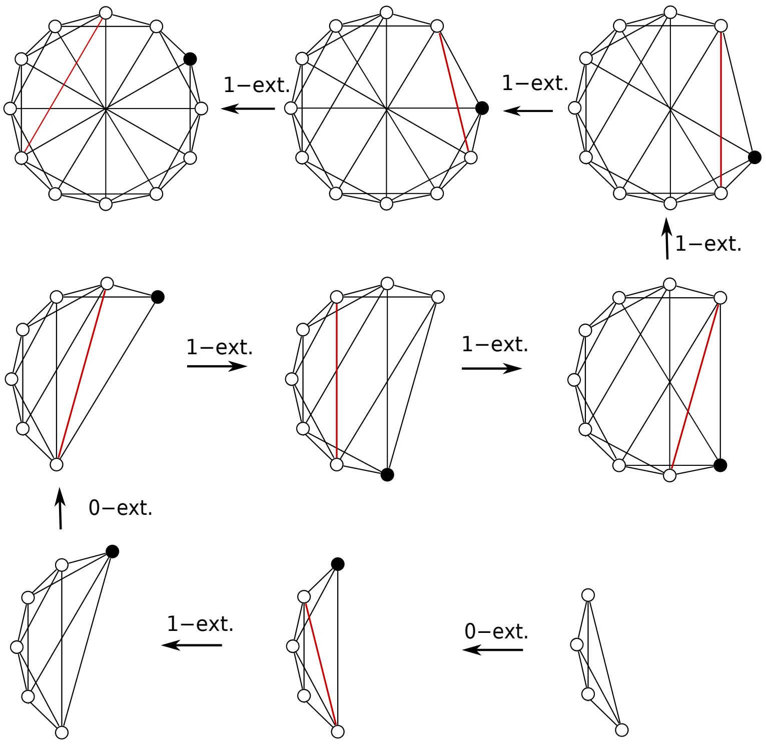


Figure 4.4: Building an edge set $H - e + f$ from K_4 by a sequence of 0-extensions and 1-extensions. New vertices are denoted by black nodes.

pendence preservation of vertex splitting operation is not a matroidal consequence of 1-extendability. Using the graph $K_{6,6}$ minus a perfect matching one can show that the independence preservation of X-replacement operation is not a matroidal consequence 1-extendability and the independence preservation of vertex splitting operation.

4.4 Intersecting submodular functions inducing abstract rigidity matroids.

Let us recall that an integer-valued function $f : 2^S \to \mathbb{Z}$ is called an *intersecting submodular function* if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for every $X, Y \subseteq S$ with $X \cap Y \neq \emptyset$. In this section when we talk about intersecting submodular functions, we mean non-decreasing intersecting submodular functions. Recall also that an intersecting submodular function $f : 2^S \to \mathbb{Z}$ induces a matroid with the collection of independent sets

$$\mathcal{I}(f) = \{I \subseteq S \mid f(J) \geq |J|, \forall J \subseteq I, J \neq \emptyset\}.$$

We say that a matroid $\mathcal{M}$ on S is induced by intersecting submodular function f if the collection of independent sets of $\mathcal{M}$ coincides with $\mathcal{I}(f)$. Note that different intersecting submodular functions can induce the same matroid.

Laman's condition can be restated as follows.

> The intersecting submodular function f defined by $f(X) = 2|V(X)| - 3$, for $\emptyset \neq X \subseteq K$, induces $\mathcal{G}_2(n)$.

A natural question is: What are necessary and sufficient conditions for an intersecting submodular function to induce $\mathcal{G}_d(n)$? We can also consider a relaxed version on the conditions for an intersecting submodular function to induce an abstract rigidity matroid.

In the following, we will show that all intersecting submodular functions inducing an abstract rigidity matroid must have the same value as the rank function of the rigidity matroid on the edge set of a complete subgraph of $K(V)$. Conversely, this condition together with the "symmetry" will ensure that the induced matroid is an abstract rigidity matroid.

Theorem 4.4.1. *Let $f : 2^K \to \mathbb{Z}$ be an intersecting submodular function that induces a d-dimensional abstract rigidity matroid $\mathcal{M}$ on K. Let $K_t \subseteq K$ be the* *edge set of a complete subgraph on t vertices for $t \geq 1$. Then $f(K_t) = dt - d(d+1)/2$ holds for $t \geq d+2$.*

We say that a function $f : 2^K \to \mathbb{Z}$ is *iso-symmetric* if $f(E) = f(F)$ whenever $E, F \subseteq K$ induce two isomorphic subgraphs.

Theorem 4.4.2. *Suppose that $f : 2^K \to \mathbb{Z}$ satisfies $f(K_t) = dt - d(d+1)/2$ for $t \geq d+2$, $f(K_t) \geq |K_t|$ for $t \leq d+1$, for the edge set K_t of any complete subgraph on t vertices of K. Suppose further that f is iso-symmetric. Then f induces a d-dimensional abstract rigidity matroid on K.*

For the sake of simplicity we will demonstrate the two theorems above for the 2-dimensional case. The same arguments work for higher dimensions.

Proof of Theorem 4.4.1.
We prove by induction on t. Let $K_4 \subseteq K$ be the edge set of a complete subgraph on 4 vertices. Since K_4 is a circuit of the abstract rigidity matroid $\mathcal{M}$ (by Theorem 4.2.2) and f induces $\mathcal{M}$ we have $f(K_4) < |K_4|$ and $f(K_4 - e) \geq |K_4 - e|$ for all $e \in K_4$. Therefore, $f(K_4) = |K_4| - 1 = 2 \times 4 - 3$ holds. Suppose that $f(K_t) = 2t - 3$ holds for some $t \geq 4$.

Claim 4.4.3. *Let $v_1, v_2 \in V(K_t)$, $v \in V(K) \backslash V(K_t)$. Then $f(K_t + vv_1 + vv_2) = f(K_t) + 2$.*

Proof. We have

$$\begin{aligned} f(K_t + vv_1 + vv_2) &\geq r_{\mathcal{M}}(K_t + vv_1 + vv_2) \\ &= r_M(K_t) + 2 \qquad \text{(by Lemma 4.2.1)} \\ &= f(K_t) + 2 \qquad \text{(by induction hypothesis).} \end{aligned}$$

Suppose on contrary that $f(K_t + vv_1 + vv_2) \geq f(K_t) + 3$. Let v_3 be a vertex in $V(K_t) - \{v_1, v_2\}$ and B_t a base of K_t. We prove that $B_t + vv_1 + vv_2 + vv_3$ is independent in $\mathcal{M}$, which has greater rank than K_{t+1}, a contradiction. We just need to prove that $f(X) \geq |X|$ for every subset X of $B_t + vv_1 + vv_2 + vv_3$. If $\{vv_1, vv_2, vv_3\} \nsubseteq X$ or $X = \{vv_1, vv_2, vv_3\}$ then evidently X is independent in $\mathcal{M}$ by Lemma 4.2.1, thus $f(X) \geq |X|$ holds. Now, if $X = X_t + vv_1 + vv_2 + vv_3$ with $\emptyset \neq X_t \subseteq B_t$, then

$$\begin{aligned} f(X) + f(B_t) &\geq f(X \cup B_t) + f(X \cap B_t) \\ &= f(B_t + vv_1 + vv_2 + vv_3) + f(X_t) \\ &\geq f(B_t) + 3 + |X_t|. \end{aligned}$$

It implies that $f(X) \geq |X|$ holds. $\square$

The following claim completes the proof of Theorem 4.4.1 for $d = 2$.

Claim 4.4.4. *Let k be an integer such that $2 \leq k \leq t$. Suppose that $v_1, \ldots, v_k \in V(K_t)$ and $v \notin V(K_t)$. Then $f(K_t + vv_1 + \cdots + vv_k) = f(K_t) + 2$ holds.*

Proof. We prove by induction on k. When $k = 2$ the statement holds by Claim 4.4.3. For $k = 3$, if $f(K_t + vv_1 + vv_2 + vv_3) \geq f(K_t) + 3$ then using the same argument as that in Claim 4.4.3 we deduce a contradiction. Thus the statement holds for $k = 3$. Now suppose that the statement holds for some $k \geq 3$. Let v_{k+1} be a vertex in $V(K_t) - \{v_1, \ldots v_k\}$, then using the intersecting submodularity of f we have

$$\begin{aligned} f(K_t + vv_1 + vv_2 + \cdots + vv_k + vv_{k+1}) + f(K_t + vv_2 + \cdots + vv_k) \\ \leq f(K_t + vv_1 + \cdots + vv_k) + f(K_t + vv_2 + \cdots + vv_{k+1}), \end{aligned}$$

which implies that $f(K_t + vv_1 + vv_2 + \cdots + vv_k + vv_{k+1}) \leq f(K_t) + 2$ and thus $f(K_t + vv_1 + vv_2 + \cdots + vv_k + vv_{k+1}) = f(K_t) + 2$. $\square$

Proof of Theorem 4.4.2.
Let $\mathcal{M}(f)$ be the matroid induced by f. Let K_t be the edge set of an arbitrary complete subgraph on t vertices of K.

Claim 4.4.5. *If $v \notin V(K_t)$ and $v_1 \in V(K_t)$ then $f(K_t + vv_1) \geq f(K_t) + 1$ holds.*

Proof. Assume on the contrary that $f(K_t + vv_1) = f(K_t)$. Then, by the iso-symmetry of f, $f(K_t + vv_i) = f(K_t)$ holds for every $v_i \in V(K_t)$. Using the intersecting submodularity of f we can easily derive that $f(K_t) = f(K_t + vv_1 + \cdots + vv_t)$, where $v_1, \ldots, v_t$ are all the vertices of K_t. Then the edge set $K_{t+1} = K_t + vv_1 + \cdots + vv_t$ of the complete subgraph on $t+1$ vertices satisfies $2(t+1) - 3 = f(K_{t+1}) = f(K_t) = 2t - 3$, which is a contradiction. $\square$

Claim 4.4.6. *If $v \notin V(K_t)$ and $v_1, v_2 \in V(K_t)$ then $f(K_t + vv_1 + vv_2) \geq f(K_t) + 2$.*

Proof. From Claim 4.4.5 we have $f(K_t + vv_1 + vv_2) \geq f(K_t) + 1$. Assume on the contrary that $f(K_t + vv_1 + vv_2) = f(K_t) + 1$. Using induction on k, the intersecting submodularity and the iso-symmetry of f we can deduce that $f(K_t + vv_1 + \cdots + vv_k) = f(K_t) + 1$ holds for every k and $v_1, \ldots, v_k \in V(K_t)$. In particular, the edge set $K_{t+1} = K_t + vv_1 + \cdots + vv_t$ of the complete subgraph on $t + 1$ vertices satisfies $2(t + 1) - 3 = f(K_{t+1}) = f(K_t) + 1 = 2t - 2$, a contradiction. $\square$

Claim 4.4.7. *A* 1*-valent* 0*-extension of an independent set of* $\mathcal{M}(f)$ *is again an independent set of* $\mathcal{M}(f)$*.*

Proof. Let B_t be a base of K_t in matroid $\mathcal{M}(f)$, $v \notin V(K_t)$ and $v_1 \in V(K_t)$. Then $f(B_t + vv_1) = f(K_t + vv_1) \geq f(K_t) + 1 = f(B_t) + 1$, by Claim 4.4.5. Let X be an arbitrary non-emty subset of B_t. By the intersecting submodularity

$$\begin{aligned} f(X + vv_1) + f(B_t) &\geq f((X + vv_1) \cup B_t) + f((X + vv_1) \cap B_t) \\ &= f(B_t + vv_1) + f(X) \\ &\geq f(B_t) + 1 + |X|. \end{aligned}$$

Thus, $f(X + vv_1) \geq |X + vv_1|$ holds. It follows that $B_t + vv_1$ is independent in the matroid $\mathcal{M}(f)$, which implies the statement of Claim 4.4.7 □

Claim 4.4.8. *A* 2*-valent* 0*-extension of an independent set in* $\mathcal{M}(f)$ *is again an independent set in* $\mathcal{M}(f)$*.*

Proof. Let B_t be a base of K_t in the matroid $\mathcal{M}(f)$, $v \notin V(K_t)$ and $v_1, v_2 \in V(K_t)$. Assume on the contrary that there exists a subset X of $B_t + vv_1 + vv_2$ such that X is a circuit in $\mathcal{M}(f)$. Then, by Claim 4.4.7, $X = X_t + vv_1 + vv_2$ with $X_t \subseteq B_t$. Then, by the intersecting submodularity of f,

$$\begin{aligned} f(X) + f(B_t) &\geq f(X \cup B_t) + f(X \cap B_t) \\ &= f(B_t + vv_1 + vv_2) + f(X_t) \\ &= f(K_t + vv_1 + vv_2) + f(X_t) \\ &\geq f(B_t) + 2 + |X_t| \qquad \text{(by Claim 4.4.6).} \end{aligned}$$

Therefore, $f(X) \geq |X|$ holds, a contradiction. □

Claim 4.4.7, Claim 4.4.8 and Theorem 4.2.2 imply that $\mathcal{M}(f)$ is a 2-dimensional abstract rigidity matroid on K, which completes the proof of Theorem 3.7 for $d = 2$. □

4.5 A potential application

We end this chapter by discussing a potential application of our results on abstract rigidity matroids.

Thomassen [103] conjectured that there exists a function $f(k)$ such that every $f(k)$-connected graph has a k-connected orientation. Jordán [68] confirmed this conjecture for $k = 2$ by showing that $f(2) \leq 18$. This upper bound is improved by Cheriyan, Durand de Gevigney and Szigeti [18] to $f(2) \leq 14$, using a similar idea. For $k \geq 3$ no upper bound has been obtained. The main idea of Jordán [68] is that a 18-connected graph contains 3 edge-disjoint 2-connect spanning subgraphs. Using these 2-connected subgraphs he deduces a 2-connected orientation of the original graph. Thus comes the question if we can pack many k-connected spanning subgraphs in a highly connected graph.

The 2-connected graphs used by Jordán are in fact 2-dimensional generically rigid graphs and the packing is obtained by using the rank formula given by the intersecting submodular function $f(F) = 2|V(F)| - 3$ which induces $\mathcal{G}_2$.

Therefore, we are interested in the question of finding a matroid such that the "rigid" graphs with respect to this matroid are k-connected, and that it possesses a "simple" inducing intersecting submodular function. k-dimensional abstract rigidity matroids may be good candidates since rigid graphs with respect to these matroids are k-connected and the sufficient condition for an intersecting submodular function to induce an abstract rigidity matroid is simple as shown in Section 4.4.

Chapter 5

Inductive constructions and decompositions of graphs

Contents

5.1 Introduction

This chapter presents combinatorial optimization results developed to solve the problem of characterizing the underlying graphs of infinitesimally rigid generic frameworks of many types. The starting point is the following fundamental result of Nash-Williams.

Theorem 5.1.1 (Nash-Williams [83]). *A graph $G = (V, E)$ can be decomposed into m edge-disjoint spanning trees if and only if $i(X) \leq m|X| - m$ for every non empty subset X of V and $|E| = m|V| - m$.*

Recall that a graph G verifying the condition in the theorem above is called an (m, m)-tight graph. An inductive construction for (m, m)-tight graphs is also implicit in [83].

In fact, a typical proof that a graph has an infinitesimally rigid realization if and only if it has a spanning tight sparse subgraph, proceeds along the following lines:

(a) Construct a rigidity matrix whose rank determines the infinitesimal rigidity of a framework;

(b) Deduce that the existence of a tight sparse subgraph is a necessary condition for infinitesimal rigidity;

(c) Use either an inductive construction, or a decomposition, of a tight sparse subgraph to demonstrate sufficiency by constructing a realization whose rigidity matrix attains the maximum rank.

An inductive construction for $(2, 3)$-sparse graphs suggested by Henneberg is used to prove Laman's fundamental result (see [10, 47, 58]. Similarly, Tay [99] used the inductive construction for (m, m)-tight graphs due to Nash-Williams [82], to show that when $m = \binom{d+1}{2}$, these graphs are exactly the underlying graphs of minimally infinitesimally rigid d-dimensional generic body-bar frameworks. Inductive techniques were also employed successfully by Katoh and Tanigawa [72] to settle the long-standing Molecular Conjecture (on rigidity of panel-hinge frameworks). The alternative approach in (c) is to use a decomposition of tight sparse graphs to give a direct construction of an infinitesimally rigid realization. Examples of this approach are Tay [98], Whiteley [107], and Jackson and Jordán [60].

The above mentioned result of Nash-Williams [82] was not motivated by rigidity theory. However, the applications mentioned above have stimulated interest in generalizing the results of Nash-Williams and Tutte.

The first way to generalize these results is to consider (m, l)-tight graphs. Fekete and Szegő [33] provide an inductive construction of (m, l)-tight graphs for $0 \leq l \leq m$. Whiteley [109] deduces a decomposition of (m, l)-tigh graph for $0 \leq l \leq m$ from matroid decomposition. Haas [51] characterizes (m, l)-tight sparse graph for $0 \leq m \leq l < 2m - 1$ in terms of an "lTm-tree decomposition", a concept rooted in Crapo's work on decomposition of Laman's graphs [26]. These decompositions are re-obtained by Streinu and Theran [96] using pebble games.

The study of frameworks with different kinds of constraints led Lee, Streinu and Theran [79] to consider a more generalized class of graphs in which different types of edges satisfy different sparsity conditions. They call these graphs *graded sparse graphs*. As an example they consider the bar-joint-slider model in which bars are represented by non-loop edges and sliders are represented by loops. They show that a graph can be realized as a rigid slider-bar-joint 2-dimensional framework if and only if it has a subgraph H which is $(2, 0)$-tight sparse and the subgraph of H induced by the non-loop edges is $(2, 3)$-sparse. In section 5.2, we provide an inductive construction as well as a decomposition for graded tight sparse graphs. The decomposition will be used in Chapter 6 to characterize different types of body-length-direction frameworks as well as to re-obtain a result of Katoh and Tanigawa on the characterization of underlying graphs of generic body-bar frameworks with bar-boundary.

We also propose a second way to generalize the result of Nash-Williams by considering $(\mathbf{b}, l)$-tight sparse graphs. These tight sparse graphs arise in the rigidity context when one considers frameworks that contain bodies of different dimensions. An inductive construction for a special instant of $\mathbf{b}, l$ was derived by Tay [98] to characterize body-rod-bar frameworks. In section 5.3, we give an inductive construction of $(\mathbf{b}, l)$-tight graphs for l not greater than the minimum value of $\mathbf{b}$. We also characterize $(\mathbf{b}, l)$-sparse graphs as resulting graphs in $(\mathbf{b}, l)$-pebble games, a generalization of (k, l)-pebble games by Lee, Streinu [78].

A third way of generalization is proposed by Katoh and Tanigawa in [73]. They regard each tree as a tree with a root r and replace the condition that each vertex is covered by all the m trees [1] with a matroidal condition on the root set of trees covering each vertex. Motivated by the study of bar-joint-slider frameworks, their result is applied successfully to characterize several other types of frameworks with boundaries (cf. Section 6.2). In Section 5.4, we derive a directed counterpart to

[1]This condition is equivalent to the condition that the m trees in the decomposition/packing are spanning.

the result of Katoh and Tanigawa. Furthermore, we show that our directed result implies their undirected result. As a consequence, we obtain a shorter proof for Katoh and Tanigawa's result.

5.2 Graded sparse graphs

5.2.1 Introduction

Let $G = (V, E)$ be a graph. An *r-grading* of E is a strictly decreasing sequence of sets $(E_1, \ldots, E_r)$ with $E = E_1 \supset E_2 \cdots \supset E_r$. A graph G with an r-grading $(E_1, \ldots, E_r)$ is called an *r-graded graph*, or simply a *graded graph* when the value of r is clear. The *grade* of an edge $e \in E$, denoted by σ_e, is the maximum number t such that $e \in E_t$. Let m be a positive integer and $\mathbf{d} = (d_1, \ldots, d_r)$ be an r-tuple of integers with $0 \leq d_1 \leq \cdots \leq d_r \leq 2m - 1$. The r-graded graph G is said to be *$(m, \mathbf{d})$-graded sparse* if the subgraph $G_t = (V, E_t)$ is (m, d_t)-sparse for every $t = 1, \ldots, r$, and *$(m, \mathbf{d})$-graded tight* if in addition it has $|E| = m|V| - d_1$. For a subset X of V, we denote by $i_t(X)$ the number of edges of E_t induced by X. The graded sparsity condition can then be written as

$$i_t(X) \leq m|X| - d_t, \qquad \text{for all non-empty } X \subseteq V \text{ and all } t = 1, \ldots, r.$$

We will need some more definitions and notations to state our main results on graded sparse graphs. A *splitting off* at a vertex v of G is the operation which deletes two edges $f = vx$ and $g = vy$ incident to v and adds a new edge $e = xy$ of grade $\sigma_e = \min\{\sigma_f, \sigma_g\}$. When f is a loop at v the splitting off just deletes f and changes the grade of g to $\min\{\sigma_f, \sigma_g\}$. We will refer to such a splitting off as a *loop replacement*. A splitting off for which neither f nor g is a loop will be referred to as a *proper splitting off*.

A *(k, ℓ)-reduction* of an $(m, \mathbf{d})$-graded tight graph G is defined as follows. We first choose a vertex v which is incident with $m+k$ edges in total, including exactly ℓ loops, for some $0 \leq k \leq m - \ell$. If $k = 0$ then we simply delete v from G. If $k \geq 1$ then we perform a sequence of ℓ loop replacements followed by k proper splitting offs at v, then delete v and the remaining $m - k - \ell$ edges incident to it. A (k, ℓ)-reduction at v is *admissible* if the resulting graph is also $(m, \mathbf{d})$-graded tight.

The inverse operation to a (k, ℓ)-reduction is called a (k, ℓ)-extension. A *(k, ℓ)-extension* of H is an r-graded graph obtained from H by the following steps.

(a) Delete k edges $e_i = x_i y_i$, $1 \leq i \leq k$, from H.

(b) Add a new vertex v and $m - k - \ell$ new edges of arbitrary grades from v to the vertices of H.

(c) For each i, $1 \leq i \leq k$, add new edges $f_i = vx_i$, $g_i = vy_i$ and ℓ_i new loops $h_{i,j}$ at v in such a way that:

(i) the minimum grade of $f_i, g_i, h_{i,1}, \ldots, h_{i,\ell_i}$ is equal to σ_{e_i};

(ii) $\ell_1 + \ell_2 + \ldots + \ell_k = \ell$;

(iii) the total number of loops of grade at least t added at v is less than or equal to $m - d_t$ for all $1 \leq t \leq r$.

In the special case when $k = 0$ we obtain a $(0, \ell)$-extension by simply adding a new vertex v, ℓ loops incident with v and $m - \ell$ edges from v to H in such a way that condition (c)(iii) above is satisfied.

Contribution: Our first result is that admissible reductions always exist, when $0 \leq d_1 \leq \cdots \leq d_r \leq m$.

Theorem 5.2.1. *Suppose that m is a positive integer and $\mathbf{d} = (d_1, \ldots, d_r)$ is a non-decreasing sequence of integers $0 \leq d_1 \leq \cdots \leq d_r \leq m$. Let G be an $(m, \mathbf{d})$-graded tight graph on at least two vertices. Then G has a vertex v which is incident with $m + k$ edges, including exactly ℓ loops, for some $0 \leq k \leq m - \ell$. Furthermore, for any such vertex v, G has an admissible (k, ℓ)-reduction at v.*

We will show that a (k, ℓ)-extension preserves the property of being $(m, \mathbf{d})$-graded tight. Combined with Theorem 5.2.1, this will imply the following inductive construction for $(m, \mathbf{d})$-graded tight graphs.

Theorem 5.2.2. *Suppose that m is a positive integer and $\mathbf{d} = (d_1, \ldots, d_r)$ a non-dereasing sequence of integers $0 \leq d_1 \leq \cdots \leq d_r \leq m$. Let G be an r-graded graph. Then G is $(m, \mathbf{d})$-graded tight if and only if G can be obtained from an $(m, \mathbf{d})$-graded tight graph on one vertex by a sequence of (k, ℓ)-extensions. Moreover, if $d_1 > 0$ then we need only (k, ℓ)-extensions with $k + \ell < m$.*

A *pseudoforest* is a graph in which each connected component contains at most one cycle. A pseudoforest is *tight* if each of its components contains exactly one cycle. (Equivalently G is a pseudoforest if it is $(1, 0)$-sparse and is a tight pseudoforest if it is $(1, 0)$-tight.) An *$(m, \mathbf{d})$-graded pseudoforest decomposition* of an r-graded graph G is a partition of $E(G)$ into m edge-disjoint pseudoforests $F_1, F_2, \ldots, F_m$ such that for each t, $1 \leq t \leq r$, and all i, $1 \leq i \leq d_t$, the restriction of F_i on E_t is a forest. Our third main result characterizes $(m, \mathbf{d})$-graded tightness in terms of $(m, \mathbf{d})$-graded pseudoforest decompositions.

Theorem 5.2.3. *Let $0 \leq d_1 \leq \cdots \leq d_r \leq m$ be integers and G an r-graded graph. Then G is $(m, \mathbf{d})$-graded tight if and only if G has an $(m, \mathbf{d})$-graded pseudoforest decomposition consisting of d_1 spanning trees and $m - d_1$ spanning tight pseudoforests.*

Theorems 5.2.1, 5.2.2 and 5.2.3 will be proved in Sections 5.2.2, 5.2.3 and 5.2.4, respectively.

Furthermore, in Section 5.2.5 we study the matroids whose independent sets are the edge sets of $(m, \mathbf{d})$-graded sparse graphs. We give a short direct proof for their matroid decomposition and thus obtain an alternative proof for the pseudoforest decomposition in Theorem 5.2.3.

Results in this section are from a joint work with Bill Jackson [66].

5.2.2 A reduction theorem for graded sparse graphs

We suppose throughout this section that

$$m \text{ is a positive integer and } \mathbf{d} = (d_1, \ldots, d_r) \text{ with } 0 \leq d_1 \leq \cdots \leq d_r \leq m. \tag{5.2.1}$$

Our aim is to prove Theorem 5.2.1. We first show that every $(m, \mathbf{d})$-graded tight graph G has a vertex which is incident with the required number of edges.

Lemma 5.2.4. *Let $G = (V, E)$ be an $(m, \mathbf{d})$-graded tight graph. Then G has a vertex v which is incident with $m + k$ edges for some $0 \leq k \leq m - i(v)$.*

Proof. Let v be a vertex of minimum degree in G. Since G is tight, we have

$$\sum_{x \in V} d(x) = 2|E| = 2m|V| - 2d_1 \leq 2m|V|.$$

Hence $d(v) \leq 2m$ and so $d(v) - i(v) \leq m + m - i(v)$. On the other hand

$$m|V| - d_1 = |E| = i(V - v) + d(v) - i(v) \leq m(|V| - 1) - d_1 + d(v) - i(v)$$

so $d(v) - i(v) \geq m$. $\square$

We next show that if v is a vertex of G with $d(v) - i(v) = m + k$ for some $1 \leq k \leq m - i(v)$, then we can construct an $(m, \mathbf{d})$-graded sparse graph from G by performing a sequence of splitting offs at v (consisting of $i(v)$ loop replacements followed by k proper splitting offs). We construct these splitting offs one at a time in such a way that we preserve $(m, \mathbf{d})$-graded sparseness as well as an additional sparsity condition at v.

We assume for the remainder of this section that

$$H = (V, E) \text{ is an } (m, \mathbf{d})\text{-graded sparse graph with grading } (E_1, \ldots, E_r) \tag{5.2.2}$$

and v is a vertex of H with

$$d(v) = m + k + c - 2s,\ i(v) = \max\{0, c - s\},\ c \geq 0,\ k \geq 1 \text{ and } 0 \leq s < k + c \leq m. \tag{5.2.3}$$

We imagine that c is the number of loops incident to v in G and that H has been obtained from G by performing s splitting offs at v. Noting that the number of edges incident to v is $d(v) - i(v)$, condition (5.2.3) implies that

$$\text{The number of edges incident to } v \text{ is strictly greater than } m - s. \tag{5.2.4}$$

In fact, if $i(v) = 0$ then the number of edges incident to v is $d(v) = m+k+c-2s = m - s + (k + c - s) > m - s$ by (5.2.3). If $i(v) > 0$ then, by (5.2.3), $i(v) = c - s$, so the number of edges incident to v is $d(v) - i(v) = m + k + c - 2s - (c - s) = m + k - s > m - s$ since $k \geq 1$. For $X, Y \subseteq V$, we denote the intersection of E_t and the set of edges from X to Y by $E_t(X, Y)$. (This set contains in particular all edges of E_t induced by $X \cap Y$.) We also use $E(X, Y)$ for $E_1(X, Y)$, $i_t(X, Y)$ for $|E_t(X, Y)|$ and $i(X, Y)$ for $|E(X, Y)|$.

We say that H is (v, s)-*good* if $i_t(X) \leq m|X| - d_t - s$ for all $X \subseteq V$ that properly contain v, and all $1 \leq t \leq r$. We will suppose henceforth that

$$H \text{ is } (v, s)\text{-good.} \tag{5.2.5}$$

Our aim is to find a splitting off at v in H such that the new graph is both $(m, \mathbf{d})$-graded sparse and $(v, s+1)$-good. We call such a splitting off *feasible*. To do this we need to consider the circumstances in which a splitting off is not feasible. A nonempty set $X \subseteq V - v$ is said to be t-*critical*, or simply *critical*, if it satisfies $i_t(X) = m|X| - d_t$ for some $1 \leq t \leq r$. A set $X \subseteq V$ is t-*crucial*, or simply *crucial*, if it properly contains v and satisfies $i_t(X) = m|X| - d_t - s$. The *grade* $\sigma(X)$ of a crucial set X is the maximum value of t for which X is t-crucial.

The following result characterizes when the splitting off operation is feasible.

Lemma 5.2.5. *Let $f = vx$ and $g = vy$ be two edges incident to v with $\sigma_f \leq \sigma_g$. Then splitting off f and g at v is feasible if and only if*
(a) no t-critical set contains both x and y for all $1 \leq t \leq \sigma_f$, and

(b) all crucial sets X have $\sigma(X) \leq \sigma_g$ and $X \cap \{x, y\} \neq \emptyset$. In addition, if $\sigma(X) > \sigma_f$, then $y \in X$.

Proof. This follows from the definitions of a feasible splitting off, and critical and crucial sets. □

It follows that we will need to analyze the structure of the families of critical and crucial sets in order to show that there exists a feasible splitting off. Our basic tool is the following result.

Lemma 5.2.6. *Suppose that $X, Y \subseteq V$ and that t, t' are integers with $1 \leq t \leq t' \leq r$. Then*

$$i_t(X) + i_{t'}(Y) + i_t(X \setminus Y, Y \setminus X) \leq i_t(X \cup Y) + i_{t'}(X \cap Y).$$

Proof. This follows easily by counting the contribution of each edge of H to both sides of the inequality. □

Lemma 5.2.7. *Suppose X is a t-critical set, Y is a t'-critical set, $t \leq t'$ and $X \cap Y \neq \emptyset$. Then $X \cup Y$ is t-critical, $X \cap Y$ is t'-critical and $i_t(X \setminus Y, Y \setminus X) = 0$.*

Proof. Since H is $(m, \mathbf{d})$ graded sparse, we have

$$\begin{aligned} i_t(X \cup Y) + i_{t'}(X \cap Y) &\leq m|X \cup Y| - d_t + m|X \cap Y| - d_{t'} \\ &= m|X| - d_t + m|Y| - d_{t'} \\ &= i_t(X) + i_{t'}(Y) \\ &\leq i_t(X \cup Y) + i_{t'}(X \cap Y) - i_t(X \setminus Y, Y \setminus X). \end{aligned}$$

Hence equality must occur everywhere and so $i_t(X \cup Y) = m|X \cup Y| - d_t$, $i_{t'}(X \cup Y) = m|X \cup Y| - d_{t'}$ and $i_t(X \setminus Y, Y \setminus X) = 0$. □

An immediate consequence of Lemma 5.2.7 is the following.

Corollary 5.2.8. *Suppose $x \in V - v$ is contained in at least one critical set. Then the union of all critical sets which contain x is a t-critical set where t is the smallest grade of all the critical sets which contain x.*

The proof of the following lemma is similar to that of Lemma 5.2.7.

Lemma 5.2.9. *Suppose that X is a t-crucial set, Y is a t'-crucial set, $t \leq t'$ and $|X \cap Y| \geq 2$. Then $X \cup Y$ is t-crucial and $X \cap Y$ is t'-crucial.*

Lemma 5.2.10. *If X is a t-critical set then $i_t(v, X+v) \leq m-s$.*

Proof. Since H is (v,s)-good, we have

$$m(|X|+1) - d_t - s \geq i_t(X+v) = i_t(X) + i_t(v, X+v) = m|X| - d_t + i_t(v, X+v).$$

Therefore, $i_t(v, X-v) \leq m-s$. $\square$

Lemma 5.2.11. *If X is a t-crucial set then $i_t(v,X) \geq m-s$ and $i_t(v, X-v) \geq 1$.*

Proof. We have

$$i_t(X-v)+i_t(v,X) = i_t(X) = m|X|-d_t-s = m|X-v|-d_t+m-s \geq i_t(X-v)+m-s$$

since H is $(m, \mathbf{d})$-graded sparse. Hence $i_t(v,X) \geq m-s$. Suppose $i_t(v, X-v) = 0$. Then $m-s \leq i_t(v,X) = i_t(v) \leq i(v) = \max\{0, c-s\}$. Since $m-s > 0$ by (5.2.3), we have $m-s \leq c-s$ and $m \leq c$. This contradicts the facts that $m \geq k+c$ and $k \geq 1$ by (5.2.3). $\square$

A crucial set X is a *minimal crucial set* if it is not properly contained in any other crucial set.

Lemma 5.2.12. *Suppose X and Y are distinct minimal crucial sets of grades t and t', respectively, where $t \leq t'$. Then $X \cap Y = \{v\}$, $E(v,V) = E_t(v,X) \cup E_{t'}(v,Y)$ and $E(v) = E_{t'}(v)$.*

Proof. The fact that $X \cap Y = \{v\}$ follows from Lemma 5.2.9 and minimality. By Lemma 5.2.11 and (5.2.3),

$$m+k+c-2s = d(v) \geq i_t(v,X)+i_{t'}(v,Y) \geq 2(m-s) = m+m-2s \geq m+k+c-2s.$$

Hence, equality must hold everywhere. In particular $d(v) = i_t(v,X) + i_{t'}(v,Y)$, which implies that $E(v,V) = E_t(v,X) \cup E_{t'}(v,Y)$ and $E(v) = E_{t'}(v)$. $\square$

Lemma 5.2.13. *There are at most two distinct minimal crucial sets in H, and the grade of a minimal crucial set is the maximum grade among all the crucial sets that contain it.*

Proof. Suppose that X_1, X_2 and X_3 are three distinct minimal crucial sets. Then $X_i \cap X_j = \{v\}$ and $X_i \cup X_j$ contains all neighbours of v for all $1 \leq i < j \leq 3$ by Lemma 5.2.12. This implies that all edges incident to v are loops, and contradicts the fact that $i(v, X_i - v) \geq 1$ by Lemma 5.2.11. The second part of the lemma follows immediately from Lemma 5.2.9. $\square$

Lemma 5.2.14. *There exists a feasible splitting off at v, and this can be taken to be a loop replacement when $i(v) \geq 1$.*

Proof. We first consider the case when $i(v) \geq 1$. Let f be a loop at v. Suppose that H has two minimal crucial sets X and Y, of grades t and t', respectively, where $t \leq t'$. Then $f \in E_{t'}(v)$ by Lemma 5.2.12 and all crucial sets have grade at most t' by Lemma 5.2.13. We may choose an edge $g \in E_{t'}(v, Y - v)$ by Lemma 5.2.11. Lemma 5.2.5 now tells us that splitting off f and g at v will be feasible. We next suppose that H has a unique minimal crucial set X, of grade t say. Then all crucial sets have grade at most t by Lemma 5.2.13 and $i_t(v, X - v) \geq 1$ by Lemma 5.2.11. Lemma 5.2.5 now tells us that splitting off f with any edge $g \in E_t(v, X - v)$ will be feasible. When H has no crucial sets, Lemma 5.2.5 implies that splitting off f with any other edge at v will be feasible. Hence we may suppose that

$$i(v) = 0.$$

Let us consider the case that H has two minimal crucial sets X and Y of grades t and t', respectively.We may choose edges $f = vx \in E_t(v, X - v)$ and $g = vy \in E_{t'}(v, Y - v)$ by Lemma 5.2.11. If there is no critical set containing both x and y, splitting off f, g at v is feasible by Lemma 5.2.5. Suppose that there is a critical set containing both x and y. Let U be the maximal one and let t'' be the minimal integer such that U is t''-critical. Note that by Corollary 5.2.8, every critical set which contains x is a subset of U and t'' is the smallest grade of all the critical sets containing x. By Lemma 5.2.10, $i_{t''}(v, U + v) \leq m - s$. On the other hand, the number of edges incident to v is strictly greater than $m - s$ by (5.2.4). Therefore, we can find an edge $h \in E(v, V - v)$ with either $\sigma_h < t''$ or $h \in E(v, V - U - v)$. This edge must belong to $E_t(v, X - v)$ or $E_{t'}(v, Y - v)$ by Lemma 5.2.12. Without loss of generality, suppose that $h \in E_{t'}(v, Y - v)$. If $\sigma_h < t''$ then splitting f, h at v is feasible by Lemma 5.2.5. If $h = vz \in E(v, V - U - v) \cap E_{t'}(v, Y - v)$, we claim that there is no critical set containing both x and z. In fact, if there is a critical set U' containing both x and z, by Lemma 5.2.7, $U \cup U'$ is a critical set properly containing x and y, which contradicts the maximality of U. Therefore, splitting off f, h at v is also feasible in this case by Lemma 5.2.5.

We now consider the case that there exists at most one minimal crucial set. If there exists a crucial set, let X be the unique minimal crucial set and t its grade. Otherwise let $X = V$ and $t = 1$. Then by Lemma 5.2.13, there is no crucial set of grade strictly greater than t. Let $f = vx$ be an edge in $E_t(v, X - v)$. If there is a critical set containing x, let U be the maximal one and suppose that

t' is the smallest integer such that U is t'-critical. Note that then there is no t'' critical set containing x with $t'' < t'$. By Lemma 5.2.10, $i_{t'}(v, U+v) \leq m-s$. On the other hand, the number of edges incident to v is strictly greater than $m-s$ by (5.2.4). Therefore, we can find an edge $h \in E(v, V-v)$ with either $\sigma_h < t'$ or $h \in E(v, V-U-v)$. In both cases, splitting off f, h at v is feasible by the maximality of U and by Lemma 5.2.5. □

5.2.3 Inductive construction of graded sparse graphs

We assume that (5.2.1) continues to hold throughout this section. Suppose that H is an $(m, \mathbf{d})$-graded tight graph and k, ℓ are non-negative integers with $k+\ell \leq m$.

Lemma 5.2.15. *Suppose H is an $(m, \mathbf{d})$-graded tight graph and G is a (k, ℓ)-extension of H. Then G is $(m, \mathbf{d})$-graded tight.*

Proof. We have $|E(G)| = |E(H)| + m = m|V(H)| - d_1 + m = m|V(G)| - d_1$. Hence it will suffice to show that G is $(m, \mathbf{d})$-graded sparse. Choose $X \subseteq V(G)$ and $t \in \{1, 2, \dots, r\}$. If $v \notin X$ then $i_t^G(X) \leq i_t^H(X) \leq m|X| - d_t$ since H is $(m, \mathbf{d})$-graded sparse. If $X = \{v\}$ then $i_t^G(X) \leq m - d_t$ by condition (c)(iii) above. Hence we may suppose that v is properly contained in X. We adopt the notation used in the earlier definition of a (k, ℓ)-extension. Let b_t be the number of edges of $\{e_1, e_2, \dots, e_k\}$ which are contained in $E_t^H(X-v)$. These edges are deleted in Step (a) and hence $i_t^G(X-v) = i_t^H(X-v) - b_t \leq m|X-v| - d_t - b_t$. We add at most $m-k-\ell$ edges of E_t at v in Step (b) and at most $2k+\ell-(k-b_t)$ edges of E_t at v in Step (c). Hence

$$i_t^G(X) \leq m|X-v| - d_t - b_t + m - k - \ell + 2k + \ell - (k - b_t) = m|X| - d_t.$$

Since this holds for all t, G is $(m, \mathbf{d})$-graded sparse. □

Proof of Theorem 5.2.2

Sufficiency follows from Lemma 5.2.15. Necessity follows by induction on $|V(G)|$ and Theorem 5.2.1. □

5.2.4 Decomposition of graded sparse graphs

In this section we give a proof of the decomposition of $(m, \mathbf{d})$-graded sparse graphs in Theorem 5.2.3 using the inductive construction. In Section 5.2.5 we give another proof of this result from matroid decomposition (Theorem 5.2.20). Since this graph decomposition is in fact equivalent to the matroid decomposition, the proof in this section serves also as an alternative proof for the matroid decomposition in Theorem 5.2.17. We will use Theorem 5.2.2 to construct this graph decomposition recursively.

Let $e = xy$ be an edge of an r-graded graph H and v be a vertex (not necessary in H). We call a *v-subdivision* of e the operation that deletes e from H and adds to H two edges $f = vx$, $g = vy$ with grades σ_f, σ_g satisfying $\sigma_e = \min\{\sigma_f, \sigma_g\}$. The edges f, g are called *subdividing edges of e*. We call a v-subdivision *proper* if both f and g are non loops, otherwise we call it *improper*. Then the (k, ℓ)-extension can be seen as a sequence of k proper v-divisions and ℓ improper v-divisions followed by the addition of $m - k - \ell$ non loop edges from v to H.

Proof of Theorem 5.2.3

We prove by induction on the number of vertices. The decomposition is trivial when $|V| = 1$. Suppose that G is obtained from H by a sequence of k proper v-subdivisions and ℓ improper v-subdivisions followed by the addition of $m-k-\ell$ non loop edges from v to H. Our construction of the decomposition of G starts with a pseudoforest decomposition $\mathcal{F}$ of H, a set $D = \{e_1, \dots, e_h\}$ of edges to be subdivided and a set A of edges which can be added to G. In fact, $A = \{f_1, g_1, \dots, f_h, g_h, a_1, \dots, a_{m-h}\}$ where f_i, g_i are subdividing edges of e_i, for $i = 1, \dots, h$. The sets $\mathcal{F}, A, D$ will be updated at each v-subdivision and edge addition. At a division of an edge e to f and g, we replace the pseudoforest $F \in \mathcal{F}$ that contains e with a pseudoforest F' by deleting e from F and adding to F some edges from A. We then remove the used edges from A and remove e from D. In fact, if the pseudoforest F has not already covered v then two edges from A are added to F, otherwise, only one is added. At an edge addition of an edge a, we simply choose an appropriate pseudoforest F from $\mathcal{F}$ and add a to F.

Throughout the remaining of this section we suppose that we have a collection of m graded pseudoforests $\mathcal{F} = \{F_1, \dots, F_m\}$, a set D of edges to be subdivided and a set A containing the subdividing edges of edges in D as well as other edges to be added. Our construction of the decomposition will decrease $|D| + |A|$.

We will need some more notation. Let $\mathcal{F}_t$ denote $\{F_{d_t+1}, \dots, F_m\}$ for $1 \le t \le r$. Set $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_{r+1} = \emptyset$. For $1 \le t \le r$, we will use A^t to denote the subset of edges in A of grade at least t and for a pseudoforest F, we also use F^t to denote the subgraph of F induced by the edges of grade at least t. When $t = r + 1$, A^t

and F^t are just empty sets. Let $t_i = \min\{t : 1 \le t \le r, i \le d_t\}$ and $t_i = r+1$ if $i > d_r$. We also set $d_{r+1} = m$. Note that, t_i is the smallest t such that $F_i \notin \mathcal{F}_t$. We restate this in the following convenient form.

$$F_i \in \mathcal{F}_t \text{ if and only if } t < t_i. \tag{5.2.6}$$

The following properties will be maintained during the construction of the pseudoforest decomposition.

(P1) For $F \in \mathcal{F} \setminus \mathcal{F}_t$, F^t contains no cycle, for $1 \le t \le r$.

(P2) $|\{F \in \mathcal{F}_t : F \text{ covers } v\}| + |\{\text{loops in } A^t\}| \le m - d_t$, for $1 \le t \le r$.

(P3) $|\{F \in \mathcal{F} : F \text{ covers } v\}| + |A| - |D| = m$.

Note that at the very beginning, when $\mathcal{F}$ is the pseudoforest decomposition of H, these conditions hold, and when the construction terminates, $|A| = |D| = 0$ hence we obtain the desired decomposition.

Suppose that at a step in the construction, a pseudoforest $F_i \in \mathcal{F}$ is replaced by a pseudoforest F_i' and we obtain a new family $\mathcal{F}' = \mathcal{F} - F_i + F_i'$. To preserve the condition (P1) for the new family $\mathcal{F}'$ we only need to make sure that

(P1') $(F_i')^{t_i}$ contains no cycle.

If $|D| + |A| = 0$ then the collection $\mathcal{F}$ is the desired decomposition. So suppose that

$$|D| + |A| > 0.$$

First consider the case when $D = \emptyset$. If there is a loop at v then let c to be one of highest grade. By condition (P2), and since $|\mathcal{F}_{\sigma_c}| = m - d_{\sigma_c}$, there is an $F_i \in \mathcal{F}_{\sigma_c}$ that does not cover v. Let $F_i' = F_i + c$, $A' = A - c$ and $D' = D$. Since F_i does not cover v, F_i' is a pseudoforest. Moreover, the only new cycle created by the operation in F' is the loop c. Note that $\sigma_c < t_i$ by (5.2.6), so $(F_i')^{t_i}$ contains no cycle. Hence (P1') holds for F_i' and therefore (P1) holds for $\mathcal{F}'$. We check condition (P2) for $\mathcal{F}'$. For $t \le \sigma_c < t_i$,

$$|\{F \in \mathcal{F}_t' : F \text{ covers } v\}| = |\{F \in \mathcal{F}_t : F \text{ covers } v\}| + 1,$$

while

$$|\{\text{loops in } (A')^t\}| = |\{\text{loops in } A^t\}| - 1,$$

thus (P2) holds for $\mathcal{F}'$. As for $t > \sigma_c$, we have $(A')^t = \emptyset$ by the maximality of σ_c, and $|\mathcal{F}'_t| = m - d_t$. Thus

$$|\{F \in \mathcal{F}'_t : F \text{ covers } v\}| + |\{\text{loops in } (A')^t\}| \leq m - d_t \quad \text{for } t > \sigma_c,$$

thus (P2) also holds for $\mathcal{F}'$ in this case. (P3) also holds for $\mathcal{F}', A', D'$ since

$$|\{F \in \mathcal{F}' : F \text{ covers } v\}| = |\{F \in \mathcal{F} : F \text{ covers } v\}| + 1, |D'| = |D|, \text{ and } |A'| = |A| - 1.$$

If there is no loop at v, let a be any edge in A and F_i be a pseudoforest that does not cover a (F_i exists by condition (P3)). Let $F'_i = F_i + a$, $A' = A - a$ and $D' = D$. (P1) holds for $\mathcal{F}'$ since no new cycle is created. (P2) also holds since $|\{\text{loops in } A^t\}| = 0$ and $|\{F \in \mathcal{F}_t : F \text{ covers } v\}| \leq |\mathcal{F}_t| = m - d_t$ for $1 \leq t \leq r$. (P3) holds for $\mathcal{F}', A', D'$ since $|\{F \in \mathcal{F}' : F \text{ covers } v\}| = |\{F \in \mathcal{F} : F \text{ covers } v\}| + 1$, $|A'| = |A| - 1$ and $|D'| = |D|$.

Now suppose that D is not empty. Take an edge $e = xy$ in D. Suppose that e is subdivided into $f = vx, g = vy$ in A. Suppose also that e is an edge of an F_i in $\mathcal{F}$.

<u>Case 1:</u> *v is not covered by F_i.*
Let $F'_i = F_i - e + f + g$, $\mathcal{F}' = \mathcal{F} - F_i + F'_i$, $D' = D - e$ and $A' = A - f - g$. First, we check condition (P1'). Suppose by contradiction that $(F_i)'^{t_i}$ contains a cycle C then, since (P1) holds for F_i, the cycle C must contain both f and g and $t_i \leq \min\{\sigma_f, \sigma_g\} = \sigma_e$. Hence $C - f - g + e$ is a cycle in $F_i^{t_i}$, a contradiction. Therefore (P1) holds for $\mathcal{F}'$. Condition (P3) also holds for $\mathcal{F}', D', A'$ since $|\{F \in \mathcal{F}' : F \text{ covers } v\}| = |\{F \in \mathcal{F} : F \text{ covers } v\}| + 1$, $|A'| = |A| - 2$ and $|D'| = |D| - 1$. If $\mathcal{F}', A'$ satisfy condition (P2) then we are done. So suppose that $\mathcal{F}', A'$ violate (P2). Since (P2) holds for $\mathcal{F}, A$, there exists $t < t_i$ such that $|\{F \in \mathcal{F}'_t : F \text{ covers } v\}| + |\{\text{loops in } (A')^t\}| > m - d_t$. Let t^* be the maximum $t < t_i$ such that this inequality holds. Then the following equality holds.

$$|\{F \in \mathcal{F}_{t^*} : F \text{ covers } v\}| + |\{\text{loops in } A^{t^*}\}| = m - d_{t^*}, \tag{5.2.7}$$

keeping in mind that the sets of loops in A and that in A' are the same as f, g are not loops.

We claim that there exists a loop in A^{t^*} of grade strictly less than t_i. Suppose by contradiction that $|\{\text{loops in } A^{t^*}\}| = |\{\text{loops in } A^{t_i}\}|$. We have,

$$|\{F \in \mathcal{F}_{t_i} : F \text{ covers } v\}| + |\{\text{loops of } A^{t_i}\}| \leq m - d_{t_i}. \tag{5.2.8}$$

Since $F_i \in \mathcal{F}_{t^*} \setminus \mathcal{F}_{t_i}$ and F_i does not cover v, we have,

$$d_{t_i} - d_{t^*} = |\mathcal{F}_{t^*} \setminus \mathcal{F}_{t_i}| > |F \in \mathcal{F}_{t^*} \setminus \mathcal{F}_{t_i} : F \text{ covers } v|. \tag{5.2.9}$$

However, from equalities (5.2.7) and (5.2.8) and the assumption $|\{\text{loops in } A^{t^*}\}| = |\{\text{loops in } A^{t_i}\}|$, we obtain

$$|F \in \mathcal{F}_{t^*} \setminus \mathcal{F}_{t_i} : F \text{ covers } v| = |\{F \in \mathcal{F}_{t^*} : F \text{ covers } v\}| - |\{F \in \mathcal{F}_{t_i} : F \text{ covers } v\}| \geq d_{t_i} - d_{t^*},$$

a contradiction to (5.2.9). Therefore our claim is true.

Let c be a loop in $A^{t^*} \setminus A^{t_i}$ of maximum grade. Note that $\sigma_c < t_i$ and hence $F'_i \in \mathcal{F}'_{\sigma_c}$ by (5.2.6). If in $F'_i - f$ the connected component containing g has a cycle then let $F''_i = F'_i - g + c$. Otherwise let $F''_i = F'_i - f + c$. Set $\mathcal{F}'' = \mathcal{F} - F_i + F''_i$, $A'' = A - f - c$ and $D'' = D - e$. We claim that $\mathcal{F}''$, A'' and D'' verify the conditions (P1), (P2), (P3). In fact, since v is already covered by F_i, $|A''| = |A| - 1$, $|D''| = |D| - 1$, (P3) obviously holds. To check (P1), first consider the case where the connected component containing g in $F'_i - f$ has a cycle. Then since F'_i is a pseudoforest, the connected component containing v in $F'_i - g$ has no cycle. Therefore, the connected component of $F''_i = F'_i - g + c$ containing v has only one cycle c and so F''_i is a pseudoforest. Recall that $\sigma_c < t_i$, we have that F''_i satisfies (P1') and hence $\mathcal{F}''$ satisfies (P1). Now consider the case where the connected component containing g of $F'_i - f$ has no cycle. Then $F''_i = F'_i - f + c$ is a pseudoforest and F''_i satisfies (P1') as $\sigma_c < t_i$. It remains to check the condition (P2) for $\mathcal{F}''$, A''. Again, suppose by contradiction that there exists t such that

$$|\{F \in \mathcal{F}''_t : F \text{ covers } v\}| + |\{\text{loops in } A''|_t\}| > m - d_t.$$

Then since $|\{F \in \mathcal{F}''_t : F \text{ covers } v\}| = |\{F \in \mathcal{F}'_t : F \text{ covers } v\}|$, and $|\{\text{loops in } A''^t\}| \leq |\{\text{loops in } A'^t\}|$, it implies

$$|\{F \in \mathcal{F}'_t : F \text{ covers } v\}| + |\{\text{loops in } A'|_t\}| > m - d_t.$$

By the maximality of t^*, we must have $t \leq t^* < t_i$. However, then $|\{\text{loops in } (A'')^t\}| = |\{\text{loops in } A^t\}| - 1$ holds, since the loop c belongs to $A^{t^*} \subseteq A^t$. Therefore, $|\{F \in \mathcal{F}''_t : F \text{ covers } v\}| + |\{\text{loops in } (A'')^t\}| = |\{F \in \mathcal{F}_t : F \text{ covers } v\}| + |\{\text{loops in } A^t\}| \leq m - d_t$, a contradiction. We conclude that (P2) also holds for $\mathcal{F}''$, A''.

Case 2: *v is already covered by F_i.*
In this case we will replace the edge $e = xy$ in F_i by one of the two edges $f = vx, g = vy$. We remove e from D and the replacing edge from D. Condition (P2) and (P3) then trivially hold.

<u>Case 2.1:</u> *The subdivision is proper.*
We will choose f or g to replace e so that F'_i is a pseudoforest and satisfies (P1').

If in $F_i - e$, one of x and y belongs to a different connected component than that contains v, without loss of generality, let it be x. Then $F'_i = F_i - e + f$ is a pseudoforest and (P1') holds since no new cycle is created. So suppose that in $F_i - e$, both x and y belong to the same connected component as v. Then the unique cycle in the connected component of F_i containing v must contain e. Therefore both $F_i - e + f$ and $F_i - e + g$ are pseudoforests. If both $(F_i - e + f)^{t_i}$ and $(F_i - e + g)^{t_i}$ contain a cycle then x and y are on the same connected component of $(F_i - e)^{t_i}$ and $\sigma_e \geq t_i$. Therefore, there exists a cycle that contains e in $F_i^{t_i}$. This contradiction to the assumption (P1) implies that either $(F_i - e + f)^{t_i}$ or $(F_i - e + g)^{t_i}$ does not contain a cycle. Without loss of generality, suppose that $F'_i = F_i - e + f$ has this property. Let $\mathcal{F}' = \mathcal{F} - F_i + F'_i$, $D' = D - e$ and $A' = A - f$. Then $\mathcal{F}', A', D'$ satisfy condition (P1), (P2) and (P3).

<u>Case 2.2:</u> *The subdivision is improper.*
Suppose that f is a loop. It is worth keeping in mind that g and e have the same ends.

If $\sigma_g = \sigma_e$ then we just put $F'_i = F_i - e + g$, $\mathcal{F}' = \mathcal{F} - F_i + F'_i$ $A' = A - g$ and $D' = D - e$. The conditions (P1), (P2), (P3) obviously hold for $\mathcal{F}', A', D'$. So let us suppose that $\sigma_g > \sigma_e$ and hence $\sigma_f = \sigma_e$.

If $\sigma_e \geq t_i$ or there is no cycle containing e in F, set $F'_i = F_i - e + g$, $A' = A - g$ and $D' = D - e$. Then F'_i is a pseudoforest and condition (P2), (P3) obviously hold. To check condition (P1'), suppose by contradiction that $(F'_i)^{t_i}$ contains cycle C'. Then the cycle must contain g since $F_i^{t_i}$ has no cycle. We obtain a cycle C in F_i from C' by replacing g with e. However, we have assumed that whether there is no cycle containing e in F_i or $t_i \leq \sigma_e < \sigma_g$, so we must have $t_i \leq \sigma_e < \sigma_g$. However, under this condition, C is a cycle in $F_i^{t_i}$, a contradiction. It means that (P1') holds for F'_i and so $\mathcal{F}', A', D'$ satisfy (P1), (P2) and (P3).

Now consider the case when $\sigma_e < t_i$ and there is a cycle containing e in F_i. Put $F'_i = F_i - e + f$, $\mathcal{F}' = \mathcal{F} - F_i + F'_i$, $A' = A - f$ and $D' = D - e$. Since there is a cycle containing e in F_i, F'_i is a pseudoforest. Since the only new cycle in F'_i is that consists of the loop f, which is of grade $\sigma_f = \sigma_e < t_i$, $(F'_i)^{t_i}$ contains no cycle, so (P1') holds for F'_i. Therefore $\mathcal{F}', A', D'$ satisfy (P1), (P2) and (P3).

□

5.2.5 Graded sparse matroids

This section is devoted to the study of $(m, \mathbf{d})$-*graded sparse matroids*, i.e. matroids whose independent sets are the edge sets of $(m, \mathbf{d})$-graded sparse graphs. The fact that these sets define a matroid was shown by Lee, Streinu and Theran [79] using the matroid circuit axioms. We will provide submodular functions that induce the $(m, \mathbf{d})$-graded sparse matroids. This gives an alternative proof for the result of [79], and also determines the rank formula for graded sparse matroids. We then use this rank formula to express an $(m, \mathbf{d})$-graded sparse matroid as a union of m smaller graded sparse matroids. This matroid decomposition results in a simple proof for Theorem 5.2.3.

We first describe a general setting for an $(m, \mathbf{d})$-graded sparse matroid. We think of the ground set as a subset of the edge set of an r-graded graph on n vertices, $K_n^{r,m}$, in which every vertex is adjacent to m loops of each grade, every pair of vertices are joined by $2m$ parallel edges of each grade, and n is a large unspecified integer. Since we work with subsets E of the edges of $K_n^{r,m}$, it is convenient to reformulate the sparsity condition for such an edge set. We say that E is $(m, \mathbf{d})$-*graded sparse* if it is the edge set of an $(m, \mathbf{d})$-graded-sparse subgraph of $K_n^{r,m}$. It is not difficult to see that an edge set E is $(m, \mathbf{d})$-graded sparse if and only if for every subset F of E, $|F^t| \leq m|V(F)| - d_t$ for all $1 \leq t \leq r$, where F^t is the set of all edges of F which have grade at least t and $V(F)$ is the set of all vertices which are incident with edges in F.

For a subset F of $E(K_n^{r,m})$, let

$$f_{m,\mathbf{d}}(F) = m|V(F)| - d_t, \tag{5.2.10}$$

where t is the minimum grade of the elements in F when $F \neq \emptyset$, and put $f_{m,\mathbf{d}}(\emptyset) = 0$. It is easy to see that $f_{m,\mathbf{d}}$ is nonnegative and nondecreasing for $\mathbf{d} = (d_1, d_2, \ldots, d_r)$ when

$$0 \leq d_1 \leq d_2 \leq \cdots \leq d_r \leq 2m - 1$$

and we will assume henceforth that this is the case. We also have the submodularity property.

Lemma 5.2.16. *The function $f_{m,\mathbf{d}}$ is an intersecting submodular function on the subsets of $E(K_n^{r,m})$.*

Proof. Let F_1, F_2 be edge sets with $F_1 \cap F_2 \neq \emptyset$. Let t_1, t_2 be the minimum grades of edges in F_1, F_2 respectively. We can suppose without loss of generality that

$t_1 \le t_2$. Then t_1 is the minimum grade of edges in $F_1 \cup F_2$, and the minimum grade of edges in $F_1 \cap F_2$ is at least t_2 . We have,

$$\begin{aligned} f_{m,\mathbf{d}}(F_1) + f_{m,\mathbf{d}}(F_2) &= m|V(F_1)| - d_{t_1} + m|V(F_2)| - d_{t_2} \\ &\ge m|V(F_1 \cup F_2)| - d_{t_1} + m|V(F_1 \cap F_2)| - d_{t_2} \\ &\ge f_{m,\mathbf{d}}(F_1 \cup F_2) + f_{m,\mathbf{d}}(F_1 \cap F_2) \end{aligned}$$

where the first inequality uses the fact that the function $F \mapsto |V(F)|$ is submodular. $\square$

Theorem 2.4.2 now implies that $f_{m,\mathbf{d}}$ induces a matroid on $E(K_n^{r,m})$. We denote this matroid by $\mathcal{M}(m, \mathbf{d})$. We next show that the independent sets of $\mathcal{M}(m, \mathbf{d})$ are the $(m, \mathbf{d})$-graded sparse edge-sets.

Theorem 5.2.17. *The edge sets of all $(m, \mathbf{d})$-graded sparse graphs form the independent sets of the matroid $\mathcal{M}(m, \mathbf{d})$.*

Proof. Suppose that E is the edge set of an $(m, \mathbf{d})$-graded sparse graph and $\emptyset \ne F \subseteq E$. Let t be the minimum grade of an edge in F. Then $|F| = |F^t| \le m|V(F)| - d_t = f_{m,\mathbf{d}}(F)$. Thus E is independent in $\mathcal{M}(m, \mathbf{d})$.

Conversely, suppose that E is independent in $\mathcal{M}(m, \mathbf{d})$. We show that the graph $(V(E), E)$ is $(m, \mathbf{d})$-graded sparse. Let $\emptyset \ne F \subseteq E$ and let t be the minimum grade of an edge in F. Then $f_{m,\mathbf{d}}(F) = m|V(F)| - d_t$. For $t' \le t$, $|F^{t'}| = |F^t| \le f_{m,\mathbf{d}}(F) = m|V(F)| - d_t \le m|V(F)| - d_{t'}$, by the $(m, \mathbf{d})$-sparseness of F and the definition of $f_{m,\mathbf{d}}$. For $t' \ge t$, $|F^{t'}| \le f_{m,\mathbf{d}}(F^{t'}) \le m|V(F)| - d_{t'}$, where the first inequality follows from the fact that F is independent in $\mathcal{M}(m, \mathbf{d})$ and the second inequality follows from the definition of $f_{m,\mathbf{d}}$, since the minimum grade of an edge in $F^{t'}$ is at least t' and the sequence $\mathbf{d}$ is nondecreasing. Therefore, $(V(E), E)$ is indeed $(m, \mathbf{d})$-graded sparse. The theorem follows. $\square$

We will refer to $\mathcal{M}(m, \mathbf{d})$ as *the $(m, \mathbf{d})$-graded sparse matroid*. Theorem 2.4.2 and Theorem 5.2.17 determine the rank formula for $\mathcal{M}(m, \mathbf{d})$.

Corollary 5.2.18. *The rank function of $\mathcal{M}(m, \mathbf{d})$ is given by*

$$r_{\mathcal{M}(m,\mathbf{d})}(E) = \min \Big\{ \sum_{j=1}^{s} f_{m,\mathbf{d}}(F_j) + |F_0| : \{F_0, F_1 \ldots, F_s\} \textit{ partitions } E \Big\}.$$

In the remainder of this section we assume that $\mathbf{d}$ satisfies the stronger hypothesis that

$$0 \le d_1 \le \cdots \le d_r \le m.$$

For $1 \le i \le m$, we define the r-tuple $\mathbf{c}^i = (c_1^i, \dots, c_r^i)$ by

$$c_t^i = \begin{cases} 1 & \text{if } i \le d_t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have $d_t = \sum_{i=1}^m c_t^i$, for all $t = 1, \dots, r$. We will show that the $(m, \mathbf{d})$-sparse matroid is the union of the m $(1, \mathbf{c}^i)$-graded sparse matroids, i.e.

$$\mathcal{M}(m, \mathbf{d}) = \mathcal{M}(1, \mathbf{c}^1) \vee \mathcal{M}(1, \mathbf{c}^2) \vee \cdots \vee \mathcal{M}(1, \mathbf{c}^m).$$

We will need the following result.

Lemma 5.2.19. *Suppose $H = (V, F)$ is a connected graph. If H is $(m, \mathbf{d})$-graded sparse and $|F| > m|V| - d_{t+1}$ for some $1 \le t \le r$ then there exists an edge e belonging to a cycle of H with $\sigma_e \le t$.*

Proof. First note that since H is $(m, \mathbf{d})$-graded sparse, $|F| \le m|V|$, and hence we must have $d_{t+1} \ge 1$. We also have $|F^{t+1}| \le m|V| - d_{t+1}$ from the sparseness of H. Since $|F| > m|V| - d_{t+1}$, there exists an edge in F of grade at most t. Suppose that every edge of grade at most t in F is a cut-edge of H. Let $F_1, \dots, F_s$ be the edge sets of the connected components of the subgraph of H induced by F^{t+1} and $F_0 = F \setminus \bigcup_{i=1}^s F_i$. Since H is connected and all edges in F_0 are cut-edges, we have $|F_0| = s - 1$. Thus

$$\begin{aligned} |F| &= |F_0| + \sum_{i=1}^{s} |F_i| \\ &\le |F_0| + \sum_{i=1}^{s} (m|V(F_i)| - d_{t+1}) \\ &= (s-1) + m|V| - d_{t+1} - (s-1)d_{t+1} \\ &\le m|V| - d_{t+1}. \end{aligned}$$

This contradiction to the assumption that $|F| > m|V| - d_{t+1}$ implies that there is an edge e with $\sigma_e \le t$ that belongs to a cycle of H. $\square$

Now we are ready to prove the following main result of this section.

Theorem 5.2.20. *Suppose $0 \le d_1 \le \cdots \le d_r \le m$. Then the matroid $\mathcal{M}(m, \mathbf{d})$ is the union of the m matroids $\mathcal{M}(1, \mathbf{c}^i)$ for $i = 1, \dots, m$.*

Proof. By definition

$$\text{if } I_i \text{ is independent in } \mathcal{M}(1, \mathbf{c}^i) \text{ then } |I_i^t| \le |V(I_i)| - c_t^i \tag{5.2.11}$$

for all $t = 1, \ldots, m$.

Let $\mathcal{M} = \mathcal{M}(1, \mathbf{c}^1) \vee \cdots \vee \mathcal{M}(1, \mathbf{c}^m)$. We need to show that $\mathcal{M} = \mathcal{M}(m, \mathbf{d})$.

First suppose that I is an independent set of $\mathcal{M}$. Then $I = I_1 \cup \cdots \cup I_m$ where I_i is an independent set in $\mathcal{M}(1, \mathbf{c}^i)$ for each $i \in \{1, \ldots, m\}$, by the definition of matroid union. Then for every t, $1 \leq t \leq r$, and for every $i = 1, \ldots, m$ we have, $|I_i^t| \leq |V(I_i)| - c_t^i$ by (5.2.11). Therefore,

$$|I^t| = \sum_{i=1}^{m} |I_i^t| \leq \sum_{i=1}^{m} (|V(I_i)| - c_t^i) \leq m|V(I)| - d_t,$$

for all $t = 1, \ldots, r$. We have a similar inequality for every subset J of I. Hence $(V(I), I)$ is indeed $(m, \mathbf{d})$-graded sparse, and hence I is independent in $\mathcal{M}(m, \mathbf{d})$ by Theorem 5.2.17.

Conversely, suppose that I is an independent set of $\mathcal{M}(m, \mathbf{d})$. We will show that I is also independent in $\mathcal{M}$. We do this by showing that the rank of I in $\mathcal{M}$ is equal to $|I|$.

By Theorem 2.3.1, to prove that $r_{\mathcal{M}}(I) = |I|$ it is sufficient to show that

$$r_{\mathcal{M}(1,\mathbf{c}^1)}(F) + \cdots + r_{\mathcal{M}(1,\mathbf{c}^m)}(F) \geq |F|$$

holds for every subset F of I.

Let $F_1, \ldots, F_k$ be the edge sets of the connected components of the subgraph H of $K_n^{r,m}$ induced by F. Then $|F| = |F_1| + \cdots + |F_k|$ and $r_{\mathcal{M}(1,\mathbf{c}^i)}(F) = r_{\mathcal{M}(1,\mathbf{c}^i)}(F_1) + \cdots + r_{\mathcal{M}(1,\mathbf{c}^i)}(F_k)$ for all $i = 1, \ldots m$. Hence we may suppose that H is connected. Let t^* be minimum grade of an edge e which belongs to a cycle C of H, where we set $t^* = 0$ if no such edge exists. Then Lemma 5.2.19 implies that $|F| \leq m|V(F)| - d_{t^*}$, taking $d_{t^*} = 0$ when $t^* = 0$. On the other hand, it is not difficult to see that

$$r_{\mathcal{M}(1,\mathbf{c}^i)}(F) = \begin{cases} |V(F)| - 1, & \text{if } 1 \leq i \leq d_{t^*}, \\ |V(F)|, & \text{otherwise.} \end{cases}$$

Since F is independent in $\mathcal{M}(m, \mathbf{d})$, this gives

$$r_{\mathcal{M}(1,\mathbf{c}^1)}(F) + \cdots + r_{\mathcal{M}(1,\mathbf{c}^m)}(F) = m|V(F)| - d_{t^*} \geq |F|.$$

The theorem now follows. $\square$

An immediate consequence of this result is that the union of two graded sparse matroids is again a graded sparse matroid.

Corollary 5.2.21. *Let* $\mathbf{d} = (d_1, \ldots, d_r)$, $\mathbf{d}' = (d'_1, \ldots, d'_r)$ *be two* r*-tuples of integers and* m, m' *be positive integers such that* $0 \leq d_1 \leq \cdots \leq d_r \leq m$ *and* $0 \leq d'_1 \leq \cdots \leq d'_r \leq m'$*. Then the union of two graded sparse matroids* $\mathcal{M}(m, \mathbf{d})$ *and* $\mathcal{M}(m', \mathbf{d}')$ *is the graded sparse matroid* $\mathcal{M}(m + m', \mathbf{d} + \mathbf{d}')$*, where* $\mathbf{d} + \mathbf{d}' = (d_1 + d'_1, \ldots, d_r + d'_r)$*, i.e.,*

$$\mathcal{M}(m + m', \mathbf{d} + \mathbf{d}') = \mathcal{M}(m, \mathbf{d}) \vee \mathcal{M}(m', \mathbf{d}').$$

The decomposition of graded sparse graphs is also a consequence of this matroid decomposition result.

Proof of Theorem 5.2.3 from matroid decomposition

Let $m, \mathbf{d}$ and $\mathbf{c}^i$, $i = 1, \ldots, m$, be defined as above. It is not difficult to see that a base F_i of $\mathcal{M}(1, \mathbf{c}^i)$ induces a pseudoforest such that the restriction of F_i on E_t is a forest if and only if $c^i_t = 1$. However, by the definition of $\mathbf{c}^i$, this is equivalent to the condition that the restriction of F_i on E_t is a forest if and only if $0 \leq i \leq d_t$. Moreover, F_i is a spanning forest if and only if $c^i_1 > 0$ and F_i is a tight spanning pseudoforest if and only if $c^i_1 = 0$. Therefore, the matroid decomposition in Theorem 5.2.17 induces our desired graph decomposition. □

5.3 $(\mathbf{b}, l)$-sparse graphs

Let V be a finite set, $\mathbf{b} : V \to \mathbb{Z}_+$, we use b_{min} to denote $\min\{b(v) : v \in V\}$. In this section we suppose that l is an integer satisfying

$$0 \le l < 2b_{min}.$$

Recall from Chapter 2 that a graph $G = (V, E)$ on V is said to be $(\mathbf{b}, l)$-sparse if $i(X) \le b(X) - l$ for every $X \subseteq V$ with $i(X) > 0$.

Contribution: This section provides an inductive construction of $(\mathbf{b}, l)$-sparse graphs for $0 \le l \le b_{min}$. We also extend the pebble game in [78] to characterize $(\mathbf{b}, l)$-sparse graphs for $0 \le l < 2b_{min}$.

5.3.1 Inductive construction of $(\mathbf{b}, l)$-sparse graphs, $0 \le l \le b_{min}$

Throughout this subsection we suppose that $0 \le l \le b_{min}$. The sparsity condition is then written simply as $i(X) \le b(X) - l$ for every $\emptyset \ne X \subseteq V$. Due to the matroidal property of the family of $(\mathbf{b}, l)$-sparse graphs, a $(\mathbf{b}, l)$-sparse graph can always be obtained from a $(\mathbf{b}, l)$-tight graph by deleting some edges. So in this subsection we are interested only in the inductive construction of $(\mathbf{b}, l)$-tight sparse graphs. The proof of our inductive construction for $(\mathbf{b}, l)$-tight sparse graphs $G = (V, E)$ is proceeded by induction on the number of vertices and follows the same line as that for $(m, \mathbf{d})$-graded tight graphs: first we choose a vertex $v \in V$ with some requirement on the number of incident edges, then we show that for this vertex there is a reduction that results in a $(\mathbf{b}, l)$-tight graph on $V - v$. This proof is relatively simpler than the one for $(m, \mathbf{d})$- tight graphs but we include it for the sake of completeness. Also, since situations in the two proofs are quite similar, we will abuse several terms already used in the previous section.

The required condition for the chosen vertex v is that $d(v) = b(v) + k + i(v)$ with $0 \le k + i(v) \le b(v)$. The existence of a vertex satisfying this condition is assured by the following lemma.

Lemma 5.3.1. *Let $G = (V, E)$ be a $(\mathbf{b}, l)$-tight sparse graph on at least two vertices. Then there exists a vertex $v \in V$ such that $d(v) = b(v) + k + i(v)$ for some $k \ge 0$ such that $0 \le k + i(v) \le b(v)$.*

Proof. First, for every $v \in V$, we have $d(v) - i(v) = i(V) - i(V - v) \ge b(V) - l - (b(V - v) - l) = b(v)$.

Suppose by contradiction that there is no vertex $v \in V$ satisfying the condition in the lemma, then for every $v \in V$, $d(v) > 2b(v)$ holds. Then

$$2|E| = \sum_{v \in V} d(v) > 2 \sum_{v \in V} b(v) = 2b(V) = 2(|E| + l),$$

a contradiction. The lemma follows. □

A reduction operation consists of splitting offs and loop deletions. A *splitting off* at a vertex v of a graph G is an operation that deletes two edge $f = vx$, $g = vy$ incident to v and adds an edge $e = xy$. If f is a loop this operation simply deletes f, we refer to this as a *loop deletion*. If neither f nor g is a loop, we call the operation a *proper splitting off*.

Let v be a vertex of a $(\mathbf{b}, l)$-tight graph G incident with c loops and $d(v) = b(v) + k + c$, for some $0 \leq k \leq m$. A *k-reduction* at v first does c loop deletions at v following by k proper splitting offs at v, then deletes all the remaining $b(v) - k$ edges. Note that when $k = c = 0$, the k-reduction simply deletes from G the vertex v and all its incident edges. A k-reduction at v is *admissible* if the resulting graph is also $(\mathbf{b}, l)$-sparse.

Suppose that H is a $(\mathbf{b}, l)$-sparse graph and v is a vertex of H with

$$\begin{gathered} d(v) = b(v) + k + c - 2s, i(v) = \max\{0, c - s\}, \\ c \geq 0, k \geq 1 \text{ and } 0 \leq s < k + c \leq b(v). \end{gathered} \tag{5.3.1}$$

We can imagine that c is the number of loops at v in G and H is obtained from G after s splitting offs. This condition implies that

$$\text{The number of edges incident to } v \text{ is strictly greater than } b(v) - s. \tag{5.3.2}$$

In fact, this number is $d(v) - i(v) = b(v) + k + c - 2s - i(v)$. If $i(v) = 0$ then $b(v) + k + c - 2s - i(v) = b(v) + k + c - 2s > b(v) - s$ since $s < k + c$ by assumption 5.3.1. If $i(v) > 0$ then $i(v) = c - s$ by assumption 5.3.1. Hence $b(v) + k + c - 2s - i(v) = b(v) + k + c - 2s - (c - s) = b(v) + k - s > b(v) - s$ since $k \geq 1$ by assumption 5.3.1.

Assumption 5.3.1 also implies that $b(v) - i(v) \geq b(v) - (c - s) \geq k + s \geq 1$. Therefore, we have

$$\text{there are non loop edges at } v. \tag{5.3.3}$$

We also say that H is *(v, s)-good* if $i(X) \leq b(X) - l$ for every $X \subseteq V$ and $i(X) \leq b(X) - l - s$ for every $X \subseteq V$ that properly contains v. In the rest of this section we will assume that

$$H \text{ is } (v, s)\text{-good.} \tag{5.3.4}$$

We will show that we can find a splitting off at v that results in a $(v, s+1)$-good graph. Such a splitting off is called *feasible*.

The two potential obstacles for a feasible splitting off are critical sets and crucial sets. A *critical set* is a set $X \subseteq V - v$ with $i(X) = b(X) - l$. A *crucial set* is a set $\{v\} \subsetneq X \subset V$ with $i(X) = b(X) - l - s$. More precisely, we have the following.

Lemma 5.3.2. *A splitting off of $f = vx, g = vy$ at v is feasible if and only if*

1. *no critical set contains both x and y, and*
2. *all crucial sets intersect the set $\{x, y\}$.*

Proof. Let H' be the graph obtained from H by splitting off f, g at v. First suppose that the splitting off of f, g at v is feasible, i.e., H' is $(v, s+1)$-good. Then for every critical set $X \subseteq V - v$, $b(X) - l = i_H(X) \leq i_{H'}(X) \leq b(X) - l$. Hence, equality must holds everywhere, in particular, $i_H(X) = i_{H'}(X)$, which implies that X does not contain both x and y. Every crucial set Y must intersect $\{x, y\}$ since, otherwise, we have $b(Y) - l - s - 1 \geq i_{H'}(Y) = i_H(Y) = b(Y) - l - s$, a contradiction.

Conversely, suppose that there is no critical set containing both x and y and all crucial sets intersect $\{x, y\}$. Let X be a subset of $V - v$. If X is not a critical set then $i_{H'}(X) \leq i_H(X) + 1 \leq b(X) - l$ holds. If X is a critical set then X does not contain both x and y, thus $i_{H'}(X) = i_H(X) = b(X) - l$ holds. Now let Y be a subset of V that properly contains v. If Y is not crucial then $i_{H'}(Y) \leq i_H(Y) \leq b(Y) - l - s - 1$ holds. If Y is a crucial set then Y intersects $\{x, y\}$, so $i_{H'}(Y) \leq i_H(Y) - 1 \leq b(Y) - l - s - 1$ holds. Therefore, splitting off f, g at v is feasible. ☐

In the following, we will investigate properties of critical sets and crucial sets. The following relations will be often of use.

$$i(X) + i(Y) + i(X \setminus Y, Y \setminus X) = i(X \cup Y) + i(X \cap (Y).$$

In particular,

$$i(X) + i(Y) \leq i(X \cup Y) + i(X \cap (Y).$$

It is also useful keeping in mind that

$$b(X) + b(Y) = b(X \cup Y) + b(X \cap Y).$$

The first result shows that the intersection and union of two intersecting critical sets are critical sets.

Lemma 5.3.3. *Let X, Y be two critical sets with $X \cap Y \neq \emptyset$. Then $X \cup Y$ and $X \cap Y$ are critical sets. Moreover, $i(X \setminus Y, Y \setminus X) = 0$.*

Proof. We have

$$\begin{aligned} b(X) - l + b(Y) - l + i(X \setminus Y, Y \setminus X) &= i(X) + i(Y) + i(X \setminus Y, Y \setminus X) \\ &= i(X \cup Y) + i(X \cap Y) \\ &\leq b(X \cup Y) - l + b(X \cap Y) - l \\ &= b(X) - l + b(Y) - l. \end{aligned}$$

Therefore equality must hold everywhere, so $i(X \cup Y) = b(X \cup Y) - l$, $i(X \cap Y) = b(X \cap Y) - l$ and $i(X \setminus Y, Y \setminus X) = 0$. The lemma follows. □

An immediate corollary of this lemma is the following.

Corollary 5.3.4. *The union of a collection of critical sets containing a common element x is a critical set.* [2]

The next result indicates that the number of edges from a critical set to v is upperbounded by $b(v) - s - i(v)$.

Lemma 5.3.5. *Let X be a critical set. Then $i(v, X + v) \leq b(v) - s$.*

Proof. Since X is critical we have, $b(X) + b(v) - l - s = b(X + v) - l - s \geq i(X + v) = i(X) + i(v, X + v) = b(X) - l + i(v, X + v)$. Hence $i(v, X + v) \leq b(v) - s$. □

The following result shows that if two crucial sets intersect in at least two vertices then their union and intersection are also crucial.

Lemma 5.3.6. *Let X, Y be crucial sets with $|X \cap Y| \geq 2$, then $X \cup Y$ and $X \cap Y$ are crucial.*

Proof. The proof is similar to that about critical sets. We include it for the sake of completeness.

$$\begin{aligned} b(X) - l - s + b(Y) - l - s &\leq i(X) + i(Y) \\ &\leq i(X \cup Y) + i(X \cap Y) \\ &\leq b(X \cup Y) - l - s + b(X \cap Y) - l - s \\ &= b(X) - l - s + b(Y) - l - s. \end{aligned}$$

Hence equality must hold everywhere, so $i(X \cup Y) = b(X \cup Y) - l - s$ and $i(X \cap Y) = b(X \cap Y) - l - s$. The lemma follows. □

[2] In fact, the following stronger assertion holds: Let $X_1, \ldots, X_t$ be critical sets such that $X_i \cap (\bigcup_{1 \leq j \leq i-1} X_j) \neq \emptyset$ for every $i = 1, \ldots, t$. Then $\bigcup_{1 \leq i \leq t} X_i$ is a critical set.

In contrast to critical sets, the number of edges from a crucial set to v is lowerbounded by $b(v) - s - i(v)$ as showed in the following result.

Lemma 5.3.7. *Let X be a crucial set. Then $i(v, X) \geq b(v) - s$. Moreover, $i(v, X - v) \geq 1$.*

Proof. We have

$$\begin{aligned} b(X) - l - s &= i(X) \\ &= i(X - v) + i(v, X) \\ &\leq b(X - v) - l + i(v, X) \\ &= b(X) - b(v) - l + i(v, X). \end{aligned}$$

Therefore $i(v, X) \geq b(v) - s$. Now suppose by contradiction that $i(v, X - v) = 0$. Then $i(v) = i(v, X) \geq b(v) - s > 0$ by assumption 5.3.1. Therefore, again by assumption 5.3.1, $i(v) = c - s$. Hence we have, $c - s \geq b(v) - s$, which implies $c \geq b(v)$, a contradiction to the assumption $b(v) \geq k + c$ and $k \geq 1$. □

The next result shows that if two crucial sets intersect at only v, then their union contains all the neighbors of v.

Lemma 5.3.8. *Let X, Y be crucial sets with $X \cap Y = \{v\}$. Then $E(v, V) = E(v, X \cup Y)$.*

Proof. We have

$$\begin{aligned} b(v) + k + c - 2s &= d(v) \quad \text{(by assumption 5.3.1)} \\ &\geq i(v, X) + i(v, Y) \quad (\text{since } X \cap Y = \{v\}) \\ &= b(v) - s + b(v) - s \quad \text{(by Lemma 5.3.7)} \\ &\geq b(v) - 2s + k + c \quad \text{(by assumption 5.3.1)} \end{aligned}$$

Therefore equality must hold everywhere, in particular, $d(v) = i(v, X) + i(v, Y)$. Thus $E(v, V) = E(v, X \cup Y)$. □

We will also need the following result.

Lemma 5.3.9. *There are at most two different (inclusionwise) minimal crucial sets in H.*

Proof. Suppose that X, Y, Z are three minimal crucial sets in H. If $|X \cap Y| \geq 2$ then, by Lemma 5.3.6, $X \cap Y$ is a crucial set, so X must coincide to Y by the minimality. So let us suppose that $X \cap Y = \{v\}$. Lemma 5.3.8 implies that $X \cup Y$ contains all the neighbors of v. By Lemma 5.3.7, $Z - v$ contains a neighbor z of v, so z must belong to $X \cup Y$. Assume without loss of generality that $z \in X$. Then $|X \cap Z| \geq 2$, thus $X \cap Z$ is a crucial set, so $X = Z$ by the minimality. The lemma follows. □

Lemma 5.3.10. *There exists a feasible splitting off at v.*

Proof. Since all crucial sets contain v, a loop deletion at v is alway feasible by Lemma 5.3.2. So if there exists a loop at v, the lemma holds. Therefore we may suppose that

$$i(v) = 0.$$

Let us first consider the case that there exist two different minimal crucial sets X and Y. Then let $f = vx$ be an edge in $E(v, X - v)$ and $g = vy$ be an edge in $E(v, Y - v)$ (their existence is guaranteed by Lemma 5.3.7). Our aim is to prove that splitting off f, g at v is feasible. Since every crucial set contains either X or Y by Lemma 5.3.9, the second condition on crucial sets of Lemma 5.3.2 is sastisfied. It remains to show that there is no critical set containing both x and y. Suppose on the contrary that U is a critical set containing both x and y. A similar proof to that of Lemma 5.3.3 shows that $i(U \setminus X, X \setminus U) = 0$, contradicting the fact that $g = vy \in E(U \setminus X, X \setminus U)$. Therefore, both conditions in Lemma 5.3.2 is satisfied, so splitting off f, g at v is feasible.

Now let us consider the case that there exists at most one minimal crucial set. If one exists, let X denote the minimal crucial set, otherwise, let $X = V$. Let $f = vx$ be an edge in $E(v, X - v)$ by Lemma 5.3.7. If there exists a critical set containing x, let U be the maximal one (U is in fact the inclusionwise maximum critical set containing x by Corollary 5.3.4). Then by Lemma 5.3.5, $i(v, U + v) \leq b(v) - s$ while $i(v, V) > b(v) - s$ by (5.3.2). Therefore, there exists an edge $g = vy \in E(v, V - v - U)$. Splitting off f, g at v is then feasible by Lemma 5.3.2. □

Now we are ready to prove the main reduction lemma.

Lemma 5.3.11 (Reduction lemma)**.** *If $G = (V, E)$ is a $(\mathbf{b}, l)$-tight graph then there exists an admissible k-reduction at a vertex v of G.*

Proof. We choose a vertex $v \in V$ incident with c loops and $d(v) = b(v) + k + c$ for some $0 \leq k \leq b(v)$ which exists by Lemma 5.3.1. If $k = 0$ we simply delete the vertex v. The obtained graph H is obviously $(\mathbf{b}, l)$-sparse and it is tight as well since $|E(H)| = |E(G)| - (d(v) - i(v)) = b(V) - l - b(v) = b(V(H)) - l$. If $k \geq 1$, we initialize the induction steps with $s = 0$. Lemma 5.3.2 shows that, as long as $s < k + c$, we can find a splitting off at v such that the obtained graph is $(v, s+1)$-good. Therefore, we can do $k + c$ feasible splitting offs and then delete the vertex v. The obtained graph H is $(\mathbf{b}, l)$-sparse by the definition of feasible splitting offs. It remains to check that H is tight. Using the tightness of G we have $|E(H)| = |E(G)| - c - k - (b(v) - k) = b(V) - l - c - k - (b(v) - k) = b(V(H)) - l$. The lemma follows. □

Next we define the reverse operation of k-reduction, that we call k-extension. Let $\mathbf{b} : V \to \mathbb{Z}_+$ an integer-valued function on a set V. A *k-extension* on a $(\mathbf{b}, l)$-sparse graph H with $V(H) \subset V$ is an operation that

1. chooses a vertex $v \in V - V(H)$;
2. delete $k \leq b(v)$ edges $e_i = x_i y_i$, $i = 1, \dots k$ from H;
3. for each $i = 1, \dots, k$, add edges vx_i, vy_i to H; then
4. adds $b(v) - k$ edges incident with v such that there are at most $b(v) - l$ loops among them.

We call this operation a *k-extension* on $e_1, \dots, e_k$ with new vertex v. Note that the number of edges increases by $b(v)$.

The main theorem of this section characterizes $(\mathbf{b}, l)$-tight graphs in term of inductive construction.

Theorem 5.3.12. *Let $\mathbf{b} : V \to \mathbb{Z}_+$ and $0 \leq l \leq \min\{b(u) : u \in V\}$. A graph $G = (V, E)$ on V is $(\mathbf{b}, l)$-tight if and only if it can be constructed from a graph on a single vertex $v_0 \in V$ with $b(v_0) - l$ loops by a sequence of k-extensions.*

Proof. The "if" part follows from Lemma 5.3.11 and by induction. We prove the "only if" part by induction. The graph on a single vertex v_0 with $b(v_0) - l$ loops is clearly $(\mathbf{b}, l)$-tight. Suppose that a graph $G = (V, E)$ is obtained from a $(\mathbf{b}, l)$-tight graph H by a k-extension on k edges $e_i = x_i y_i$ for $i = 1, \dots, k$ with the vertex $v \in V - V(H)$. We check the $(\mathbf{b}, l)$-tightness of G. For a subset X of $V - v$, $i_G(X) \leq i_H(X) \leq b(X) - l$ by the sparseness of H. If $X = \{v\}$, $i_G(v) \leq b(v) - l$ by the condition on the number of added loops. For a subset X of V that contains v, if X contains both ends x_i, y_i of an edge e_i, then when comparing to $E_H(X)$, in $E_G(X)$ two edges vx_i, vy_i are added while an edge e_i is deleted. So the total contribution of the deletion of e_i and addition of vx_i, vy_i to $E_G(X)$ is at most one. The contribution of any other added edge to $E_G(X)$ is also at most one. Therefore, $i_G(X) \le i_H(X - v) + b(v) \le b(X - v) - l + b(v) = b(X) - l$ holds. Hence G is $(\mathbf{b}, l)$-sparse. Furthermore, since H is tight, $|E(G)| = |E(H)| + b(v) = b(V(H)) - l + b(v) = b(V(G)) - l$. The theorem follows. $\square$

5.3.2 $(\mathbf{b}, l)$-pebble games

In this subsection, we introduce $(\mathbf{b}, l)$-pebble games which generalizes (m, l)-pebble games by Lee and Streinu [78]. We show that $(\mathbf{b}, l)$-sparse graphs can be characterized using $(\mathbf{b}, l)$-pebble games.

Let V be a finite set of vertices, $\mathbf{b} : V \to \mathbb{Z}_+$ and $0 \le l \le 2b_{min}$. Let $G = (V, E)$ be an undirected graph on V. In a $(\mathbf{b}, l)$-pebble game, each vertex v of V is provided with $b(v)$ pebbles. We consider an unoriented edge uv of G and try to orient it using the following rules.

1. *Add-edge move*: If there are totally at least $l + 1$ pebbles at u and v and p is a pebble at u, then put p on uv and orient uv from u to v. We say that uv is *covered* by p.

2. If there are totally at most l pebbles at u and v, then try to collect at least $l + 1$ pebbles at u and v using *pebble-slide moves* which will be described later.

3. If it is impossible to collect at least $l + 1$ pebbles at u and v then discard the edge uv.

A *pebble-slide move* concerns an oriented edge uv (oriented from u to v). Let p be the pebble placed on uv and p' a pebble at v. A pebble-slide move puts p back to u, places p' on uv and reorients uv from v to u. Note that in a pebble game, a pebble stays either in its original vertex or on an edge adjacent to this vertex and once an edge is oriented, it remains an oriented edge (althought may be reoriented to the inverse direction). At the end of the game, we obtain a subgraph H of G whose edges are all the edges of G that are oriented by the pebble game. We say that H is constructed by a $(\mathbf{b}, l)$-pebble game. The $(\mathbf{b}, l)$-sparseness of H follows from the following lemma. At some stage in the pebble game, for a subset U of V, let *peb*(U) denote the total number of pebbles at all the vertices of U, *span*(U) denote the number of oriented edges with both ends in U and *out*(U) denote the number of oriented edges with tail in U and head in $V \setminus U$. If $U = \{v\}$, we simply use $peb(v), span(v)$ and $out(v)$ to refer to $peb(U), span(U)$ and $out(U)$. We denote by D the directed graph obtained at stage of a game.

Lemma 5.3.13. *During the pebble game, the following conditions always hold on D for every $v \in V$ and for every $U \subseteq V$.*

1. $peb(v) + span(v) + out(v) = b(v)$;
2. $peb(U) + span(U) + out(U) = b(U)$;
3. $peb(U) + out(U) \geq l$ *if* $l \leq b_{min}$ *or* $|U| \geq 2$;
4. $span(U) \leq b(U) - l$ *if* $l \leq b_{min}$ *or* $|U| \geq 2$.

Proof. The first two equalities follows from the fact that in a pebble game, a pebble always stays either in its original vertex or on an edge oriented out from this vertex. Hence $peb(v) + span(v) + out(v)$ is the total number of pebbles originally placed at v and so is equal to $b(v)$. Similarly, $peb(U) + span(U) + out(U)$ is the total number of pebbles originally placed at the vertices of U and so is equal to $b(U)$. The last inequality follows easily from the second equality and the third inequality. Now we prove the third inequality by induction on the number of moves. At the beginning, all the pebbles are placed at the vertices, so $peb(U) + out(U) = b(U) \geq l$ if $l \leq b_{min}$ or $|U| \geq 2$. It is sufficient to show that after each move, this condition always holds.

First consider an add-edge move that orients an unoriented edge uv from u to v. Then by definition, before the move $pev(u) + peb(v) \geq l + 1$, so after the move $pev(u) + peb(v) \geq l$ holds. Hence if both u, v are in U we have $peb(U) + out(U) \geq l$. If at most one of u, v belongs to U then this move does not decrease $out(U)$, and if $peb(U)$ is decreased by this move it is because $u \in U$, $v \notin U$ and one pebble from u is put on uv, which decreases $peb(U)$ by one. However, in this case, the move increases $out(U)$ by one. Therefore the condition holds after an add-edge move.

Now consider a pebble-slide move on an oriented edge uv (oriented from u to v.) If both u, v are in U or both u, v are not in U then the move does not change $peb(U) + out(U)$, so the condition holds after this pebble-slide move. If $u \in U$ and $v \in V \setminus U$ then the move decreases $out(U)$ by one in reorienting uv, but it increases $peb(U)$ by one by pushing back a pebble on uv to u. So in this case, $peb(U) + out(U)$ remains the same. If $u \in V - U$ and $v \in U$, the move decreases $peb(U)$ by one but increases $out(U)$ by one. So $peb(U) + out(U)$ remains the same in this case too. Therefore, the condition holds after a pebble-slide move. The lemma follows. $\square$

An immediate consequence of Lemma 5.3.13 is the following.

Corollary 5.3.14. *The resulting graph in* $(\mathbf{b}, l)$*-pebble game is a* $(\mathbf{b}, l)$*-sparse graph.*

The next result shows that if we play the $(\mathbf{b}, l)$-pebble game on a $(\mathbf{b}, l)$-sparse graph G then no edge is discarded.

Lemma 5.3.15. *Let* H *be the underlying graph of the directed graph obtained at a stage of the* $(\mathbf{b}, l)$*-pebble game on a graph* G*. Suppose that an unoriented edge* uv *is then considered. Then* uv *is discarded by the pebble game if and only if* uv *is in the closure of* $E(H)$ *in the* $(\mathbf{b}, l)$*-count matroid on* G*.*

Proof. If uv is not discarded by the $(\mathbf{b}, l)$-pebble game then it is added to H. so Corollary 5.3.14 implies that $H + uv$ is $(\mathbf{b}, l)$-sparse. Therefore uv is not in the closure of $E(H)$ in the $(\mathbf{b}, l)$-count matroid on G.

Now suppose that uv is discarded by the pebble game. Let D be the directed graph obtained at the stage just before we discard uv and after all the possible pebble-slide moves to collect pebbles at u and v. Let $Reach(u)$ and $Reach(v)$ denote the set of vertices reachable from u, v, respectively, in D and let $X = Reach(u) \cup Reach(v)$. Since no more pebble can be collected to u and v, there is no pebble on every vertex x of X other than u, v, otherwise, we can do pebble-slide moves on the dipath from, say, u to x to collect one more pebble at u and v. Hence $peb(X) = peb(u) + peb(v) \leq l$. Moreover, $out(X) = 0$. Therefore, condition 2 of Lemma 5.3.13 implies that $i_H(X) = span(X) = b(X) - peb(u) - peb(v) \geq b(X) - l$. Since H is $(\mathbf{b}, l)$-sparse, $i_H(X) = b(X) - l$ holds. Hence $H + uv$ is not $(\mathbf{b}, l)$-sparse, so uv is in the closure of $E(H)$ in the $(\mathbf{b}, l)$-count matroid on G. □

Therefore we obtained the following characterization of $(\mathbf{b}, l)$-sparse graphs in term of pebble games.

Theorem 5.3.16. *A graph is* $(\mathbf{b}, l)$*-sparse if and only if it is obtained by a* $(\mathbf{b}, l)$*-pebble game.*

5.4 Packing of matroid-based arborescences

5.4.1 Introduction

Recall from Section 2.2 that an out-arborescence (resp., in-arborescence) is a directed graph in which each vertex has in-degree (resp., out-degree) at most 1 and there is exactly one vertex r with in-degree (resp., out degree) 0. Namely, an out-aborescence is a tree where all edges are oriented away from a root node while an in-arborescence is a tree where all edges are oriented toward a root node. The problem of packing arborescences has important applications in many practical problems such as evacuation, commodity, broadcasting,... For example, in evacuation situation (tsunami, earthquake, fire,...), an in-arborescence represents roads used by refugees while an out-arborescence can represent the roads use by emergency vehicles. Since each road has a limited capacity, it is preferable to have several disjoint paths from each place to the safe place, where comes the necessity of considering packing of arborescences. In broadcast networks or commodity networks, informations or commodities are also sent along arcs of arborescences. The existence of a packing of arborescences satisfying some required conditions makes sure that the informations or commodities are sent without interference.

From a theoretical view point, the following two problems are fundamental.

1. Given a digraph (a network with pre-determined orientation of edges), discern whether there exists a certain number of arc-disjoint arborescences satifying some conditions.

2. Given an undirected graph, discern whether one can orient the edges of the graph such that in the oriented graph there exists a certain number of arc-disjoint arborescences satisfying some conditions. This problem reduces to the problem of discerning whether there exists a certain number of edge-disjoint trees satisfying the corresponding conditions.

The earliest fundamental results on these problems are due to Nash-Williams, Tutte and Edmonds.

Theorem 5.4.1 (Nash-Williams [82], Tutte [104])**.** *A graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if the inequality*

$$e_G(\mathcal{P}) \geq k|\mathcal{P}| - k$$

holds for every partition $\mathcal{P}$ of V.

Theorem 5.4.1 is the dual of Theorem 5.1.1

Theorem 5.4.2 (Edmonds [31])**.** *Let $D = (V, A)$ a digraph and $r \in V$. The digraph D contains k arc-disjoint spanning out-arborescences rooted at r if and only if*

$$\rho_D(X) \geq k \quad \textit{for every nonemty } X \subseteq V - r.$$

Frank pointed out that the undirected result of Nash-Williams and Tutte can be obtained easily from its directed counterpart of Edmonds through an orientation result.

Theorem 5.4.3 (Frank [39])**.** *Let $G = (V, E)$ be an undirected graph and r a vertex of G. There exists an orientation D of G such that $\rho_D(X) \geq k$ for every nonempty $X \subseteq V - r$ if and only if $e_G(\mathcal{P}) \geq k|\mathcal{P}| - k$ for every partition $\mathcal{P}$ of V.*

The above result of Edmonds is generalized in many ways, among them we can name results on packing of branchings, packing of Steiner arborescences[3]. One of the notable generalizations is the following by Kamiyama, Katoh and Takizawa. For a digraph $D = (V, A)$ and a vertex $r \in V$, let $Reach(r)$ denote the set of all vertices v reachable from r, i.e., there exists a directed path from r to v in D. In the remainder of this section we will simply use "arborescence" to refer to "out-arborescence".

Theorem 5.4.4 (Kamiyama, Katoh and Takizawa [71])**.** *Let $D = (V, A)$ be a digraph and $R = \{r_1, \ldots, r_t\}$ a multiset of elements in V. There exist k arc-disjoint arborescences $T_1, \ldots, T_k$ in D such that T_i is rooted at r_i and spans $Reach(r_i)$ if and only if*

$$\rho_D(X) \geq p(X) \quad \textit{for every nonempty } X \subset V,$$

where $p(X)$ denotes the number of $i \in \{1, \ldots, t\}$ such that $r_i \notin X$ and there exists a directed path from r_i to an element in X.

Fujishige generalizes this results for packing arc-disjoint arborescences spanning convex sets. A *convex set* in a digraph $D = (V, A)$ is a set $U \subseteq V$ such that every dipath from u to v with $u, v \in U$ lies completely in U. Note that, for each $i \in \{1, \ldots, t\}$, $Reach(r_i)$ is obviously a convex set.

[3]The problem of discerning whether a directed graph contains k edge-disjoint Steiner arborescences is NP-hard in general, but it is polynomial when some condition on the in-degree of terminals set is satisfied [11]

Theorem 5.4.5 (Fujishige [41]). *Let $D = (V, A)$ be a digraph, $R = \{r_1, \ldots, r_t\}$ a multiset of elements of V and $U_1, \ldots, U_t$ convex sets with $r_i \in U_i$ for $i = 1, \ldots, t$. There exist arc-disjoint arborescences $T_1, \ldots, T_t$ rooted at $r_1, \ldots, r_t$ such that T_i spans U_i, for $i = 1, \ldots, t$, if and only if*

$$\rho_D(X) \geq p_R^{(U_1,\ldots,U_k)}(X) \quad \textit{for all nonempty } X \subset V$$

where $p_R^{(U_1,\ldots,U_k)}(X)$ denotes the number of $i \in \{1, \ldots, t\}$ such that $U_i \cap X \neq \emptyset$ and $r_i \notin X$.

We generalize the result of Edmonds in another direction. Suppose that $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$ is a finite set and $D = (V, A)$ is a digraph and $\pi : \mathsf{S} \to V$ is a map. We can think of π as a placement of elements of S on vertices of D. An S-rooted arborescence of D is a pair (T, s) where T is an arborescence of D rooted at a vertex $r \in V$ with $\pi(\mathsf{s}) = r$. We will also call s the root of T, S the root set of (D, π) and we say that T is rooted at s. When S is clear from the context we simply call an S-rooted arborescence a *rooted-arborescence*.

Let $\mathcal{M}$ be a matroid on S. We call the quadruple $(D, \mathcal{M}, \mathsf{S}, \pi)$ a *matroid-based rooted-digraph*. A *matroid-based packing of arborescences* is a set $\{(T_1, \mathsf{s}_1), \ldots, (T_t, \mathsf{s}_t)\}$ of pairwise arc-disjoint S-rooted arborescences of D such that for each $v \in V$, the set $\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ forms a base of $\mathcal{M}$ (Figure 5.1). We denote $S_X = \pi^{-1}(X)$ for $X \subseteq V$ and $S_v = S_{\{v\}}$ for $v \in V$.

Our main result in this section is the following.

Theorem 5.4.6. *A matroid-based rooted-digraph $(D, \mathcal{M}, \mathsf{S}, \pi)$ has a matroid-based packing of arborescences if and only if the following conditions are satisfied.*

1. *S_v is independent for every $v \in V$.*

2. *For every nonempty subset $X \subset V$, the inequality $\rho_D(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$ holds.*

A map π satisfying condition 1 in Theorem 5.4.6 is said to be $\mathcal{M}$-*independent*. A quadruple $(D, \mathcal{M}, S, \pi)$ satisfying condition 2 in this theorem is said to be *rooted-connected*.

Our result is in fact motivated by the study of the rigidity of frameworks, or more precisely, by the work of Katoh and Tanigawa on matroid-based packing of trees, which takes it motivation from the study on infinitesimal rigidity of frameworks with boundaries (see Section 6.2). Let us take a brief look at their result.

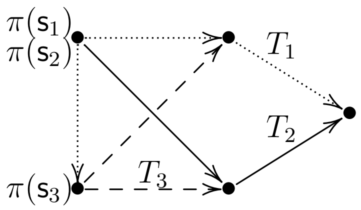


Figure 5.1: A matroid-based packing of rooted-arborescences where the set of the independent sets of the matroid on $\mathsf{S} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\}$ is $2^{\mathsf{S}} \setminus \mathsf{S}$.

Let $G = (V, E)$ be a graph, $\mathcal{M}$ a matroid on a finite set $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$ and $\pi : \mathsf{S} \to V$ a map. The quadruple $(G, \mathcal{M}, S, \pi)$ is then called a *matroid-based rooted-graph.* An S-rooted tree of G is a pair (T, s) where T is a tree of G and $\mathsf{s} \in \mathsf{S}$ with $\pi(\mathsf{s}) \in V(T)$. The element s is then called the *root* of (T, s) and the set S is called the *root set* of G. A matroid-based packing of rooted-trees of $(G, \mathcal{M}, S, \pi)$ is a set $\{(T_1, \mathsf{s}_1), \dots, (T_t, \mathsf{s}_t)\}$ of pairwise edge-disjoint S-rooted trees of G such that for each $v \in V$, the set $\{\mathsf{s}_i : v \in V(T_i)\}$ forms a base of $\mathcal{M}$. The following undirected counterpart of Theorem 5.4.6 is slightly stronger than that stated in [73] but in fact is implicit in [73].

Theorem 5.4.7 (Katoh and Tanigawa [73])**.** *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted graph. There exists a matroid-based packing of rooted-trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$ if and only if the following conditions are satisfied.*

1. *S_v is independent in $\mathcal{M}$ for each $v \in V$, and*
2. *$e_G(\mathcal{P}) \geq r_{\mathcal{M}}(\mathsf{S})|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(\mathsf{S}_X)$, for all partition $\mathcal{P}$ of V.*

A map π satisfying condition 1 in Theorem 5.4.6 is said to be *$\mathcal{M}$-independent.* A quadruple $(G, \mathcal{M}, \mathsf{S}, \pi)$ satisfying condition 2 in this theorem is said to be *partition-connected.*

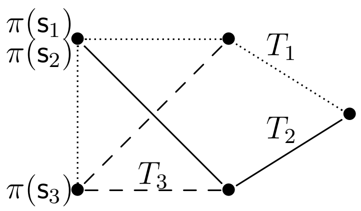


Figure 5.2: A matroid-based packing of rooted-trees where the set of the independent sets of the matroid on $\mathsf{S} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\}$ is $2^{\mathsf{S}} \setminus \mathsf{S}$.

In the same manner as Theorem 5.4.1 is obtained from its directed counter part, Theorem 5.4.2, Theorem 5.4.7 follows from our Theorem 5.4.6 through the following general orientation result of Frank.

Theorem 5.4.8 (Frank [38])**.** *Let $G = (V, E)$ be a graph and $h : 2^V \to \mathbb{Z}_+$ an intersecting supermodular non-negative non-increasing set-function. There exists an orientation D of G such that $\rho_D(X) \geq h(X)$ for all non-empty $X \subset V$ if and only if for every partition $\mathcal{P}$ of V,*

$$e_G(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} h(X).$$

Proof of Theorem 5.4.7 from Theorem 5.4.6

First suppose that there exists a matroid-based packing $\{(T_1, \mathsf{s}_1), \ldots, (T_t, \mathsf{s}_t)\}$ of rooted-trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$. Let D be an orientation of G where each rooted-tree (T_i, s_i) becomes a rooted-arborescence (T'_i, s_i). Then $\{(T'_1, \mathsf{s}_1), \ldots, (T'_t, \mathsf{s}_t)\}$ is a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. By Theorem 5.4.6, π is $\mathcal{M}$-independent and $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected and hence, by Theorem 5.4.8, $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected.

Now suppose that π is $\mathcal{M}$-independent and $(G, \mathcal{M}, \mathsf{S}, \pi)$ is partition-connected. By Theorem 5.4.8, there exists an orientation D of G such that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. Then, by Theorem 5.4.6, applied to the function $h(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X)$, there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ which provides, by forgetting the orientation, a matroid-based packing of rooted-trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$. □

In fact, Katoh and Tanigawa deduced Theorem 5.4.7 from its dual form given below. We show that Theorem 5.4.7 also implies Theorem 5.4.9.

Theorem 5.4.9 (Katoh and Tanigawa [73])**.** *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-graph. Let $\mathcal{M}$ be of rank k with rank function $r_{\mathcal{M}}$. Then $(G, \mathcal{M}, \mathsf{S}, \pi)$ admits a matroid-based rooted-tree decomposition if and only if π is $\mathcal{M}$-independent, $|E| + |\mathsf{S}| = k|V|$ and $|E(X)| \leq k|X| - k + r_{\mathcal{M}}(\mathsf{S}_X) - |\mathsf{S}_X|$ for all non-empty $X \subseteq V$.*

Proof. We first prove the necessity. The $\mathcal{M}$-independence of π is obviously necessary since each vertex is always covered by the trees rooted at elements s placed at that vertex. Let D be the orientation of G where each rooted-tree in the decomposition is oriented out from its root. Then in each arborescence, each vertex has in-degree 1 except to the root which has in-degree 0. Moreover, each vertex is covered by exactly k arc-disjoint arborescences, and there are $|\mathsf{S}|$ arborescences in total. So the total number of in-degrees of all vertices in D is $k|V|-|\mathsf{S}|$ which is also $|E|$. Therefore $|E|+|\mathsf{S}|=k|V|$ holds. Now for each vertex set $X\subseteq V$, the total number of in-degrees of all vertices in X is $\sum_{v\in X}\rho_D(v)=k|X|-|\mathsf{S}_X|$. Moreover, a vertex v in X can not be covered by more than $r_{\mathcal{M}}(\mathsf{S}_X)$ arborescences rooted in vertices of X, thus v must be covered by at least $k-r_{\mathcal{M}}(\mathsf{S}_X)$ arborescences rooted at vertices in $V\setminus X$. Hence, there are at least $k-r_{\mathcal{M}}(\mathsf{S}_X)$ arcs entering X. Therefore

$$|E(X)|=\sum_{v\in X}\rho_D(v)-\rho_D(X)\leq k|X|-k+r_{\mathcal{M}}(\mathsf{S}_X)-|\mathsf{S}_X|$$

holds.

Now suppose that the conditions hold. For every partition $\mathcal{P}$ of V, by the inequality applied for $X\in\mathcal{P}$ and by $|E|+|\mathsf{S}|=k|V|$, we have

$$\begin{aligned}
e_G(\mathcal{P}) &= |E|-\sum_{X\in\mathcal{P}}|E(X)| \\
&\geq |E|-\sum_{X\in\mathcal{P}}(k|X|-k+r_{\mathcal{M}}(\mathsf{S}_X)-|\mathsf{S}_X|) \\
&= k|\mathcal{P}|-\sum_{X\in\mathcal{P}}r_{\mathcal{M}}(\mathsf{S}_X).
\end{aligned}$$

Hence $(G,\mathcal{M},\mathsf{S},\pi)$ is partition-connected. Then, since π is $\mathcal{M}$-independent, Theorem 5.4.7 implies that $(G,\mathcal{M},\mathsf{S},\pi)$ admits a matroid-based packing of rooted-trees which, by $|E|+|\mathsf{S}|=k|V|$, must be a matroid-based rooted-tree decomposition of $(G,\mathcal{M},\mathsf{S},\pi)$. □

Contribution and organization: In Section 5.4.2 we provide a proof of our main result in this chapter, Theorem 5.4.6. This proof is short and relatively simple. Therefore, as discussed above, we obtain a short alternative proof for Theorem 5.4.7 and hence Theorem 5.4.9 of Katoh and Tanigawa. In Section 5.4.3 we offer a polyhedral description for the matroid-based packing of rooted-arborescences. Section 5.4.4 considers the algorithmic aspect of our packing problem. We show that the problem of discerning the existence of a matroid-based packing of rooted-arborescences is in P and the corresponding optimization problem also can be solved in polynomial time. Lastly, Section 5.4.5 discusses related problems and a recently extended result based on our result.

Results in this section are from a joint work with Olivier Durand de Gevigney and Zoltán Szigeti [28].

5.4.2 Proof of the main theorem

First we prove the necessity of the conditions.

Proof of necessity in Theorem 5.4.6

Suppose that there exists a matroid-based packing $\mathcal{T} = \{(T_1, \mathsf{s}_1), \ldots, (T_t, \mathsf{s}_t)\}$ of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. Let v be an arbitrary vertex of V and X a vertex set containing v. Since the root set of all rooted-arborescences covering v is a base of $\mathcal{M}$, the root set of all rooted-arborescences of $\mathcal{T}$ covering v with root in X is an independent set, in particular, S_v is independent. Hence, there are at most $r_{\mathcal{M}}(X)$ arborescences of $\mathcal{T}$ with root in X that cover v. Therefore, there are at least $r_{\mathcal{M}}(S) - r_{\mathcal{M}}(X)$ rooted-arborescences of $\mathcal{T}$ covering v which has root in $V \setminus X$. In each of these rooted-arborescences, there is at least one arc entering X. Since these rooted-arborescences are arc-disjoint, the number of arc entering X is at least $r_{\mathcal{M}}(S) - r_{\mathcal{M}}(X)$, that is $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. $\square$

To prove the sufficiency, let us introduce the following definitions. A vertex set X is called *tight* if $\rho_D(X) = r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)$. For vertex sets X and Y, we say that Y *dominates* X if $\mathsf{S}_X \subseteq \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_Y)$. Note that since, for $\mathsf{Q} \subseteq \mathsf{S}$, $\mathrm{cl}_{\mathcal{M}}(\mathrm{cl}_{\mathcal{M}}(\mathsf{Q})) = \mathrm{cl}_{\mathcal{M}}(\mathsf{Q})$, domination is a transitive relation. We say that an arc uv is *bad* if v dominates u, otherwise it is *good*. Note that only good arcs are potential candidates for arcs of rooted-arborescences in a matroid-based packing of rooted-arborescences.

Claim 1. *Suppose that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. Let X be a tight set and v a vertex of X.*

(a) *If Y is a tight set that contains v, then $X \cap Y$ and $X \cup Y$ are tight. Moreover, if $\mathsf{s} \in \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_Y)$, then $\mathsf{s} \in \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_{X \cap Y})$.*

(b) *If no good arc exists in $D[X]$, then v dominates X.*

Proof. (a) We have

$$\begin{aligned}
\rho_D(X) + \rho_D(Y) &= r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) + r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_Y) \\
&\leq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(S_X \cup \mathsf{S}_Y) + r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X \cap \mathsf{S}_Y) \\
&\leq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(S_{X\cup Y}) + r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}) \\
&\leq \rho_D(X \cup Y) + \rho_D(X \cap Y) \\
&\leq \rho_D(X) + \rho_D(Y).
\end{aligned}$$

The first equility holds by the tightness of X and Y, the second inequality holds by the submodularity of $r_{\mathcal{M}}$, the fourth holds by the rooted connectedness of $(D, \mathcal{M}, \mathsf{S}, \pi)$ and the last holds by the submodularity of ρ_D. Therefore we have equality everywhere and hence $X \cup Y$, $X \cap Y$ are tight. We also have

$$r_{\mathcal{M}}(\mathsf{S}_X) + r_{\mathcal{M}}(\mathsf{S}_Y) = r_{\mathcal{M}}(\mathsf{S}_X \cup \mathsf{S}_Y) + r_{\mathcal{M}}(\mathsf{S}_X \cap \mathsf{S}_Y)$$

holds.

If $\mathsf{s} \in \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_Y)$ then

$$\begin{aligned}
r_{\mathcal{M}}(\mathsf{S}_{X\cup Y}) + r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}) &= r_{\mathcal{M}}(S_X) + r_{\mathcal{M}}(\mathsf{S}_Y) \\
&= r_{\mathcal{M}}(\mathsf{S}_X \cup \mathsf{s}) + r_{\mathcal{M}}(\mathsf{S}_Y \cup \mathsf{s}) \\
&\geq r_{\mathcal{M}}(\mathsf{S}_X \cup \mathsf{S}_Y \cup \mathsf{s}) + r_{\mathcal{M}}((\mathsf{S}_X \cap \mathsf{S}_Y) \cup \mathsf{s}) \\
&\geq r_{\mathcal{M}}(S_{X\cup Y} \cup \mathsf{s}) + r_{\mathcal{M}}(\mathsf{S}_{X\cap Y} \cup \mathsf{s}) \\
&\geq r_{\mathcal{M}}(\mathsf{S}_{X\cup Y}) + r_{\mathcal{M}}(\mathsf{S}_{X\cap Y}).
\end{aligned}$$

Therefore, again, equality holds everywhere, in particular, $r_{\mathcal{M}}(\mathsf{S}_{X\cap Y} \cup \mathsf{s}) = r_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$ holds, which means that $\mathsf{s} \in \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$.

(b) Let us denote by Y the set of vertices from which v is reachable in $D[X]$. We show that v dominates Y and Y dominates X and then, since domination is transitive, (b) follows.

For all $y \in Y$, there exists a directed path $y = v_l, \ldots, v_1 = v$ from y to v in $D[X]$. Since no good arc exists in $D[X]$, $\mathsf{S}_y = \mathsf{S}_{v_l} \subseteq \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_{v_{l-1}}) \subseteq \cdots \subseteq \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_{v_1}) = \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_v)$. Hence $\mathsf{S}_Y = \bigcup_{y\in Y} \mathsf{S}_y \subseteq \mathrm{cl}_{\mathcal{M}}(\mathsf{S}_v)$ and v dominates Y.

By the definition of Y, every arc of D that enters Y enters X as well. Then, by the rooted-connectedness of $(D, \mathcal{M}, \mathsf{S}, \pi)$, the tightness of X and the monotonicity of $r_{\mathcal{M}}$, we have $r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_Y) \leq \rho_D(Y) \leq \rho_D(X) = r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \leq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_Y)$. Thus equality holds everywhere and Y dominates X. □

Now we can prove the main result.

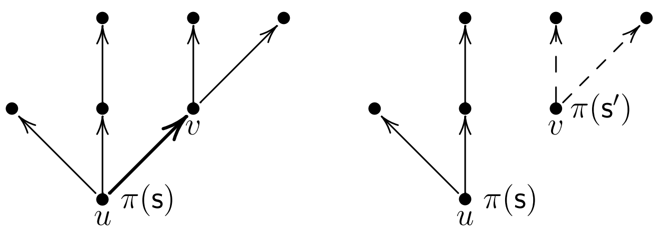
Proof of sufficiency in Theorem 5.4.6

We prove by induction on the number of good arcs.

First consider the case when no good arc exists. Let T_i be the arborescence containing only on vertex $r_i = \pi(\mathsf{s}_i)$ for $i = 1, \dots, t$. Then $\{(T_i, \mathsf{s}_i) : i = 1, \dots, t\}$ forms a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$. To prove this it is sufficient to show that S_v is a base of $\mathcal{M}$ for all $v \in V$. Indeed, let X_v be the set of vertices from which v is reachable in D. Then $\rho_D(X_v) = 0$ holds. The rooted-connectedness of $(D, \mathcal{M}, \mathsf{S}, \pi)$ implies that $r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) = 0$. However, since there are only bad arc in $(D, \mathcal{M}, \mathsf{S}, \pi)$, $r_{\mathcal{M}}(\mathsf{S}_{X_v}) = r_{\mathcal{M}}(\mathsf{S}_v)$. Therefore $r_{\mathcal{M}}(S_v) = r_{\mathcal{M}}(S)$ holds. Combined with the $\mathcal{M}$-independence of π we obtain that S_v is a base of $\mathcal{M}$.

Now suppose that at least one good arc exists.

For a good arc $uv \in A$ and $\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)$ we define a new matroid-based rooted digraph as follows. We set $D' = D - uv$, $\mathsf{S}' = \mathsf{S} \cup \mathsf{s}'$ where s' is a new element. We extend $\mathcal{M}$ to a matroid $\mathcal{M}'$ on S' by defining s' as an element parallel to s. Lastly, we obtain a placement π' of S' in V from π by placing the new element s' at v.



in D in D'

Figure 5.3: Changing rooted-arborescences.

A matroid-based packing of arborescences $\mathcal{T}'$ of $(D', \mathcal{M}', \mathsf{S}', \pi')$ will provide a matroid-based packing of arborescences $\mathcal{T}$ for $(D, \mathcal{M}, \mathsf{S}, \pi)$ as follows. Let

$$(T'', \mathsf{s}) = (T \cup T' \cup uv, \mathsf{s}),$$
$$\mathcal{T} = \mathcal{T}' \cup \{(T'', \mathsf{s})\}) \setminus \{(T, \mathsf{s}), (T', \mathsf{s}')\}.$$

Since s and s' are parallel in $\mathcal{M}'$, the rooted-arborescences (T, s) and (T', s') of $\mathcal{P}'$ are vertex disjoint, so (T'', s) is a rooted-arborescence. Then $\mathcal{T}'$ is a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$.

Since D' has less good arcs than D if π' is $\mathcal{M}'$-independent and $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected then $(D', \mathcal{M}', \mathsf{S}', \pi')$ has a matroid-based packing of arborescences by induction. Since the $\mathcal{M}'$-independence of π' is trivial from the $\mathcal{M}$-independence of π and the fact that $\mathsf{s} \in \mathsf{S}_u - \mathrm{cl}(\mathsf{S}_v)$, for the existence of a matroid-based packing of arborescences in $(D', \mathcal{M}', \mathsf{S}', \pi')$, by induction hypothesis it is sufficient to find a good arc uv and $\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)$ such that $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected.

Assume that such a good arc does not exist. Let $uv \in A$ be a good arc and $\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)$. Then since $(D', \mathcal{M}', \mathsf{S}', \pi')$ is not rooted-connected, there exists $\emptyset \neq X_\mathsf{s} \subset V$ such that $\rho_{D'}(X_\mathsf{s}) < r_\mathcal{M}(\mathsf{S}) - r_{\mathcal{M}'}(\mathsf{S}'_{X_\mathsf{s}})$. Hence, by the rooted-connectedness of $(D, \mathcal{M}, \mathsf{S}, \pi)$ and the monotonicity of $r_{\mathcal{M}'}$,

$$\begin{aligned}
\rho_{D'}(X_\mathsf{s}) + 1 &\geq \rho_D(X_\mathsf{s}) \\
&\geq r_\mathcal{M}(\mathsf{S}) - r_\mathcal{M}(\mathsf{S}_{X_\mathsf{s}}) \\
&\geq r_\mathcal{M}(\mathsf{S}) - r_{\mathcal{M}'}(\mathsf{S}'_{X_\mathsf{s}}) \\
&\geq \rho_{D'}(X_\mathsf{s}) + 1.
\end{aligned}$$

So equality holds everywhere and thus uv enters X, X_s is tight in $(D, \mathcal{M}, \mathsf{S}, \pi)$ and $\mathsf{s} \in \mathrm{cl}_\mathcal{M}(\mathsf{S}_{X_\mathsf{s}})$. Hence, by Claim 1(a), $X = \cup_{\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)} X_\mathsf{s}$ is tight and, by $v \in X$, $\mathsf{S}_u = (\mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)) \cup (\mathsf{S}_u \cap \mathrm{cl}(\mathsf{S}_v)) \subseteq \mathrm{cl}(\mathsf{S}_X) \cup \mathrm{cl}(\mathsf{S}_X) = \mathrm{cl}(\mathsf{S}_X)$. So we have proved that

$$\text{every good arc } uv \text{ enters a tight set } X \text{ that dominates } u. \tag{5.4.1}$$

Among all pairs (uv, X) satisfying (5.4.1) choose one with X minimal. Since X dominates u but v does not dominate u, v does not dominate X. Then, by Claim 1(b), there exists a good arc $u'v'$ in $D[X]$. Then, by (5.4.1), $u'v'$ enters a tight set Y that dominates u'. By $v' \in X \cap Y$, the tightness of X and Y, $u' \in X$, $\mathsf{S}_{u'} \subseteq \mathrm{cl}_\mathcal{M}(\mathsf{S}_Y)$ and Claim 1(a), we have that $X \cap Y$ is tight and $\mathsf{S}_{u'} \subseteq \mathrm{cl}_\mathcal{M}(\mathsf{S}_{X \cap Y})$. Since the good arc $u'v'$ enters the tight set $X \cap Y$ that dominates u' and $X \cap Y$ is a proper subset of X (since $u' \in X \setminus Y$), this contradicts the minimality of X. $\square$

5.4.3 Polyhedral description

In this section we provide a polyhedron describing the matroid-based packings of rooted-arborescences.

We need the following general result of Frank [37].

Theorem 5.4.10 (Frank [37]). *Let $D = (V, A)$ be a digraph, $h : 2^V \to \mathbb{Z}_+$ a non-negative intersecting supermodular set-function such that $\rho_D(X) \geq h(X)$ for every $X \subseteq V$. Then the polyhedron defined by the following linear system is integer:*

$$\begin{aligned} 1 \geq x(a) \geq 0 \qquad & \text{for all } a \in A, \\ x(R_D^-(X)) \geq h(X) \qquad & \text{for all non-empty } X \subseteq V. \end{aligned}$$

The following theorem is a corollary of Theorems 5.4.6 and 5.4.10.

Theorem 5.4.11. *Let $(D = (V, A), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph where $\mathcal{M}$ is of rank k with rank function $r_{\mathcal{M}}$ and suppose that π is $\mathcal{M}$-independent. The convex hull $P_{\mathcal{M},D}$ of characteristic vectors of the arc sets of the matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ is given by the system*

$$1 \geq x(a) \geq 0 \qquad \text{for all } a \in A, \tag{5.4.2}$$

$$x(R_D^-(X)) \geq k - r_{\mathcal{M}}(\mathsf{S}_X) \qquad \text{for all non-empty } X \subseteq V, \tag{5.4.3}$$

$$x(A) = k|V| - |\mathsf{S}|. \tag{5.4.4}$$

In particular, there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if the polyhedron $P_{\mathcal{M},D}$ is not empty.

Proof. By Theorem 5.4.10, the polyhedron P determined by the subsystem (5.4.2) and (5.4.3) is integer. For every x in this polyhedron, we have

$$x(A) = \sum_{v \in V} x(R_D^-(v)) \geq \sum_{v \in V} (k - r_{\mathcal{M}}(\mathsf{S}_v)) \geq \sum_{v \in V} (k - |\mathsf{S}_v|) = k|V| - |\mathsf{S}|. \tag{5.4.5}$$

Therefore, (5.4.4) is a valid inequality for this polyhedron P and hence $P_{\mathcal{M},D}$ is a face of P. Thus $P_{\mathcal{M},D}$ is integer. A characteristic vector of a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ obviously belongs to $P_{\mathcal{M},D}$. Conversely, if x is a vertex of $P_{\mathcal{M},D}$ then it is integer. Let $A' = \{a \in A : x(a) = 1\}$ and $D' = (V, A')$. Then the matroid-based rooted digraph $(D', \mathcal{M}, \mathsf{S}, \pi)$ is rooted-connected. Together with the assumption that π is $\mathcal{M}$-independent, from Theorem 5.4.6 we deduce that $(D', \mathcal{M}, \mathsf{S}, \pi)$ has a matroid-based packing of arborescences whose set of arcs is A'. Therefore x is the characteristic vector of this packing. The theorem follows. □

5.4.4 Algorithmic aspects

In this section we assume that a matroid is given by an oracle for the rank function, that is, when we give the matroid oracle a subset $X \in \mathsf{S}$, it will give back the rank of X. The following theorem is a corollary of Theorems 2.4.1 and 5.4.6.

Theorem 5.4.12. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph. A matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ or a vertex v certifying that π is not $\mathcal{M}$-independent or a vertex set X certifying that $(D, \mathcal{M}, \mathsf{S}, \pi)$ is not rooted-connected can be found in polynomial time.*

Proof. By the submodularity of $\rho_D(X) + r_{\mathcal{M}}(\mathsf{S}_X)$, Theorem 2.4.1, using the oracle on $\mathcal{M}$ and Theorem 5.4.6, we can either find a set violating the condition of rooted-connectedness or a vertex certifying that π is not $\mathcal{M}$-independent or certify that there exists a matroid-based packing of rooted-arborescences.

In the latter case, a matroid-based packing of rooted-arborescences can be found in polynomial time following the proof of Theorem 5.4.6. Using the oracle, one can test whether an arc is bad or good. When an arc uv is good, for each $s \in \mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)$, determine in polynomial time whether $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected using the submodularity of $\rho_{D'}(X) + r_{\mathcal{M}'}(\mathsf{S}'_X)$, the oracle for the rank function $r_{\mathcal{M}'}$ (that is easily computed from $r_{\mathcal{M}}$) and Theorem 2.4.1. The proof of Theorem 2.4.1 shows that either all arcs are bad or we find a good arc uv and $\mathsf{s} \in \mathsf{S}_u \setminus \mathrm{cl}(\mathsf{S}_v)$ such that $(D', \mathcal{M}', \mathsf{S}', \pi')$ is rooted-connected. In the first case, $\{(v, \mathsf{s}) : v \in V, \mathsf{s} \in \mathsf{S}_v\}$ is the required packing. In the second case, it leads to the computation of a matroid-based packing of rooted-arborescences in $(D', \mathcal{M}', \mathsf{S}', \pi')$ where D' contains less arcs than D. □

By the submodularity of $x(R^-_D(X)) + r_{\mathcal{M}}(\mathsf{S}_X)$ and by Theorem 2.4.1, $P_{\mathcal{M},D}$ can be separated in polynomial time. It is a well-known result by Grötschel, Lovász and Schrijver [49] that the separation problem and optimization problem are polynomial-time equivalent. Therefore, Theorem 5.4.12 leads to the following consequence.

Theorem 5.4.13. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted-digraph and c a cost function on the set of arcs of D. If there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ then one of minimum cost can be found in polynomial time.*

5.4.5 Further remarks

Theorem 5.4.6 motivates the following extension. Given a matroid-based rooted-digraph $(D, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathcal{M}$ has rank function $r_{\mathcal{M}}$ and a bound $\mathbf{b} : V \to \mathbb{Z}$, an $(\mathcal{M}, \mathbf{b})$*-packing of rooted-arborescences* is a set $\{(T_1, \mathsf{s}_1), \dots, (T_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of pairwise arc-disjoint rooted-arborescences such that $r_{\mathcal{M}}(\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}) \geq b(v)$ for all $v \in V$. What is the necessary and sufficient condition for the existance of an $(\mathcal{M}, \mathbf{b})$-packing of rooted-arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$? When the function $\mathbf{b}$ is constant, i.e., $b(v) = b$ for all $v \in V$, by truncating the matroid $\mathcal{M}$ to a matroid of rank b, then using Theorem 5.4.6, one can derive a characterization of matroid-based rooted-digraphs admitting an $(\mathcal{M}, b)$-packing of rooted-arborescences. On the other hand, for general $\mathbf{b}$, the problem turns out to be NP-complete since it contains the disjoint Steiner arborescences problem, that is to find 2 arc-disjoint r-arborescences both covering a specified subset of vertices.

Basing on the same proof technique, Csaba Király recently extends our result to maximal independent packing of arborescences. For a subset $X \subseteq V$ let $P(X)$ denote the set of vertices u such that there is a directed path from u to a vertex in X, note that $P(X)$ contains X is as a subset. When $X = \{v\}$ we simply write $P(v)$. A *maximal independent packing of arborescences* of $(D, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(T_1, \mathsf{s}_1), \ldots, (T_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of pairwise arc-disjoint S-rooted arborescences of D such that, for each $v \in V$, the set $\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ is independent in $\mathcal{M}$ and of size $r_{\mathcal{M}}(\mathsf{S}_{P(v)})$.

Theorem 5.4.14 (Cs. Király [74]). *A matroid-based rooted-digraph $(D, \mathcal{M}, \mathsf{S}, \pi)$ has a maximal independent packing of arborescences if and only if π is $\mathcal{M}$-independent and*

$$\rho_D(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$$

holds for each non-empty $X \subseteq V$.

This result also generalizes the result of Kamiyama, Katoh and Takizawa (Theorem 5.4.4). Moreover, Cs. Király also pointed out that Theorem 5.4.5 in turn can be obtained easily from Theorem 5.4.4 and therefore these two theorems are infact equivalent.

Chapter 6

Infinitesimal rigidity

Contents

6.1 Introduction

Infinitesimal rigidity plays a central role in rigidity theory. One important reason is that it is more tractable than other rigidity properties. Given a framework, its infinitesimal rigidity can be determined by calculating the rank of a rigidity matrix while determining whether the framework is locally rigid, globally rigid or universally rigid seems difficult [1, 91]. Moreover, infinitesimal rigidity implies local rigidity in general, and when we restrict ourselves to (linearly) generic frameworks, infinitesimal rigidity coincides with local rigidity [9, 65], note that almost all frameworks are (linearly) generic. Infinitesimal rigidity is also highly involved in the study of other rigidity properties. For example, Theorem 3.2.6 states that a stress matrix of maximal rank is a certificate for the global rigidity of an algebraically generic bar-joint framework. But in most cases, it is not easy to find this certificate for a generic framework. A result of Connelly and Whiteley [25] shows that if $(G, \mathbf{p})$ (not necessarily generic) is infinitesimally rigid in $\mathbb{R}^d$ and has a stress matrix of maximal rank then G is generically globally rigid in $\mathbb{R}^d$. This property is extremely useful in proving the global rigidity preservingness of certain operations on graphs. Furthermore, infinitesimal rigidity is also important in the neighborhood stability of other rigidity properties such as globally linkedness of pairs of vertices [63].

This chapter contains results on infinitesimal rigidity of frameworks with mixed constraints. In Section 6.2 we begin by reviewing body-bar frameworks, the first model where a complete characterization of infinitesimal rigidity of generic frameworks is obtained. We then consider body-bar frameworks with bar-boundary, i.e, frameworks where some bodies are linked to an external enviroment with bars. We re-obtain a characterization of infinitesimal rigidity of generic frameworks of this type using our decomposition of graded tight graphs in Section 5.2.

Section 6.3 introduces the study of body-length-direction frameworks, where several types of bodies that allow different types of motions are considered. In addition, beside distance constraints, these frameworks are also subject to direction constraints. Characterization of infinitesimal rigidity for different types of frameworks are obtained.

Section 6.4 considers the so-called direction-length frameworks, an extended model of bar-joint frameworks, where we concern also with direction constraints between vertices. Characterizing infinitesimal rigidity and global rigidity of these frameworks has application in Computer-Aided-Design and network localization.

Complete characterization is obtained by B. Servatius and Whiteley for infinitesimal rigidity in dimension 2 [94] and partial results for global rigidity are obtained by Jackson and Jordán [61]. We introduce extension operations for these frameworks and investigate the infinitesimal rigidity preservingness of these operations in general dimension. Chapter 7 deals with the global rigidity preservingness of these operations.

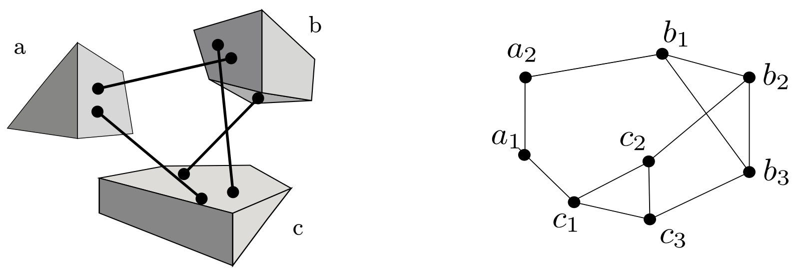


Figure 6.1: A body-bar structure and its corresponding bar-joint graph.

6.2 Body-bar frameworks and body-bar frameworks with bar-boundary

6.2.1 Body-bar frameworks

A body-bar structure is a structure constituting of solid bodies kept together by solid bars. The bodies allow translations and rotations as their motions. A body-bar structure can be modeled by a body-bar framework which is a pair $(G, \mathbf{p})$ where G is a (multi)graph without loops on the body set and $\mathbf{p}$ is an embedding of the bars. This framework can be converted to a bar-joint framework by converting each extremity of bars to a joint and adding a complete graph on the set of joints on the same body. Figure 6.1 illustrates a body-bar structure and the corresponding bar-joint graph. However, while the problem of finding a combinatorial characterization for underlying graphs of infinitesimally rigid generic bar-joint frameworks is one central open problem in rigidity theory, the same problem for body-bar frameworks is much easier.

Let $G = (V, E)$ be a graph without loops. A *realization of G as a body-bar framework* is a map $\mathbf{p}$ which associates each edge $e = uv \in E$ with a pair of points $p_u^e, p_v^e \in \mathbb{R}^d$. The actual shapes and positions of bodies in V are not of significance[1]. We may think as that each vertex $u \in V$ is assigned a d-dimensional body B_u and $p_u^e \in B_u$ for all edges e incident to u.

An *infinitesimal motion* of a body-bar framework $(G, \mathbf{p})$ is a map $\mathbf{q}$ that assigns to each body $u \in V$ a pair (A_u, t_u) where A_u is a $d \times d$ skew-symmetric real matrix and t_u a vector in $\mathbb{R}^d$ such that, for each edge $e = uv \in E$

$$\langle (A_u p_u^e + t_u) - (A_v p_v^e + t_v), p_u^e - p_v^e \rangle = 0$$

[1]In fact, in bar-joint frameworks, for the purpose of studying infinitesimal rigidity, only the line passing through p_u^e, p_v^e is of significance, but not these two points themselves.

(c.f. Section 2.6).

For each $u \in V$, let w_u be the $\binom{d}{2}$-dimensional vector associated with the skew-symmetric matrix A_u as defined in Section 2.5. Using equation (2.5.1) we can rewrite this equality as

$$\langle (p^e_u - p^e_v) \vee p^e_u \, , w_u \rangle - \langle (p^e_u - p^e_v) \vee p^e_v \, , w_v \rangle + \langle p^e_u - p^e_v \, , t_u - t_v \rangle = 0.$$

Note that $(p^e_u - p^e_v) \vee p^e_u = p^e_u \vee p^e_v$ and $(p^e_u - p^e_v) \vee p^e_v = p^e_u \vee p^e_v$, so the equality becomes

$$\langle p^e_u \vee p^e_v \, , w_u - w_v \rangle + \langle p^e_u - p^e_v \, , t_u - t_v \rangle = 0.$$

For $p, p' \in \mathbb{R}^d$ let $T(p, p') = (p, 1) \vee (p', 1) \in \mathbb{R}^{\mathtt{D}}$ where $(p, 1), (p', 1)$ denote the $(d+1)$-dimensional vectors obtained by adding 1 to p, p' as the $(d+1)$-th coordinates and $\mathtt{D} = \binom{d+1}{2}$. Put $q_u = (w_u, t_u)$. The above equality can be rewritten as

$$\langle T(p^e_u, p^e_v) \, , q_u - q_v \rangle = 0.$$

Therefore, an infinitesimal motion of $(G, \mathbf{p})$ can be regarded as an assignment to each body $u \in V$ of a vector $q_u \in \mathbb{R}^{\mathtt{D}}$ such that

$$\langle T(p^e_u, p^e_v) \, , q_u - q_v \rangle = 0 \quad \text{for all } e = uv \in E, u, v \in V. \tag{6.2.1}$$

We can also regard $\mathbf{q}$ as a $\mathtt{D}|V|$-dimensional vector obtained by putting q_u for $u \in V$ consecutively.

An infinitesimal motion of a body-bar framework $(G, \mathbf{p})$ is *trivial* if $q_u = q_v$ for all $u, v \in V$. A body-bar framework $(G, \mathbf{p})$ is said to be *infinitesimally rigid* if all its infinitesimal motions are trivial.

The *rigidity matrix* $R(G, \mathbf{p})$ of a body-bar framework $(G, \mathbf{p})$ is an $|E| \times \mathtt{D}|V|$ matrix defined as follows. Each edge of E indexes one row of $R(G, \mathbf{p})$ while each vertex of v indexes $\mathtt{D}$ columns of $R(G, \mathbf{p})$ in such a way that

- the block indexed by an edge $e = uv \in E$ and the vertices u, v are $R_{e,u} = T(p^e_u, p^e_v)$ and $R_{e,v} = -R_{e,u}$, respectively.
- all the other entries are 0's.

Then equation (6.2.1) implies that $\operatorname{Ker} R(G, \mathbf{p})$ is the space of all infinitesimal motions of $(G, \mathbf{p})$. We deduce that $(G, \mathbf{p})$ is infinitesimally rigid if and only if $\dim \operatorname{Ker} R(G, \mathbf{p})$ is $\mathtt{D}$, which is the dimension of the space of all trivial infinitesimal motions. Equivalently speaking, $(G, \mathbf{p})$ is infinitesimally rigid if and only if the rank of $R(G, \mathbf{p})$ is equal to $\mathtt{D}|V| - \mathtt{D}$.

A d-dimensional body-bar framework $(G, \mathbf{p})$ is said to be *generic* if every submatrix of $R(G, \mathbf{p})$ attains the maximum rank among all d-dimensional realizations of G. Then it is not difficult to see that if $(G, \mathbf{p})$ has an infinitesimally rigid realization then all its generic realizations are infinitesimally rigid. Moreover, similarly to the case of bar-joint frameworks, almost all d-dimensional realizations of a graph G as body-bar frameworks are generic.

Let $\mathbf{e}^0 = (0, \ldots, 0) \in R^d$ and for $i = 1, d$ let $\mathbf{e}^i$ denote the ith unit vector of $\mathbb{R}^d$. Put $T_{h,k} = T(\mathbf{e}^h, \mathbf{e}^k)$.

Lemma 6.2.1. *The family of* $\mathtt{D}$ *vectors* $\{T_{h,k} : 0 \leq h < k \leq d\}$ *is a base of* $\mathbb{R}^{\mathtt{D}}$*.*

Proof. It is easy to check the following fact on the $\{i, j\}$ entry of $T_{h,k}$.

For $h = 0$, $1 \leq k \leq d$

$$T_{h,k}\{i, j\} = \begin{cases} -1 & \text{if } i = k, j = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq h < k \leq d$,

$$T_{h,k}\{i, j\} = \begin{cases} -1 & \text{if } i = k, j = d + 1 \\ 1 & \text{if } i = h, j = k \text{ or } i = k, j = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that scalars $\alpha_{h,k}$ satisfy

$$\sum_{0 \leq h < k \leq d} \alpha_{h,k} T_{h,k} = 0, \tag{6.2.2}$$

we show that they are all zero. Indeed, if $j \neq d + 1$ then $T_{h,k}\{i, j\} \neq 0$ if and only if $h = i$ and $k = j$. Evaluating the equation (6.2.2) for an entry $\{i, j\}$ with $j \neq d + 1$, we obtain $\alpha_{i,j} = 0$ for all $1 \leq i < j \leq d$. Thus equation (6.2.2) becomes

$$\sum_{1 \leq k \leq d} \alpha_{0,k} T_{0,k} = 0.$$

Recall that $T_{0,k}\{i, j\} \neq 0$ if and only if $k = i, j = d + 1$, by evaluating this equation for entries $\{i, d+1\}$, $1 \leq i \leq d$, we get $\alpha_{0,i} = 0$ for all $1 \leq i \leq d$. Therefore, we have shown that $\alpha_{h,k} = 0$ for all h, k, $0 \leq h < k \leq d$. That is, $\{T_{h,k} : 0 \leq h < k \leq d\}$ is linearly independent. The lemma follows.[2] $\square$

[2] In fact, we can take any $p_0, \ldots, p_d$ affinely independent. It is a well known fact that then $\{(p_h, 1) \vee (p_k, 1) : 0 \leq h < k \leq d\}$ is linearly independent in $\mathbb{R}^{\mathtt{D}}$.

The following theorem by Tay characterizes graphs that can be realized as infinitesimally rigid body-bar frameworks in $\mathbb{R}^d$.

Theorem 6.2.2 (see [101],[99],[107]). *A graph $G = (V, E)$ can be realized as an infinitesimally rigid body-bar framework in dimension d if and only if G contains $\mathtt{D}$ edge-disjoint spanning trees.*

Proof. We present a proof using ideas from [60] [73].

If G can be realized as an infinitesimally rigid d-dimensional body-bar framework $(G, \mathbf{p})$ then $\operatorname{rank} R(G, \mathbf{p}) = \mathtt{D}|V| - \mathtt{D}$. Let H be the spanning subgraph of G whose edge set indexes a maximal linearly independent set of rows in $R(G, \mathbf{p})$. We show that H is $(\mathtt{D}, \mathtt{D})$-tight. Since $|E(H)| = \mathtt{D}|V| - \mathtt{D}$, it will suffice to show that H is $(\mathtt{D}, \mathtt{D})$-sparse. Let $X \subseteq V$ and $F \subseteq E$ be the edge set of H induced by X. Let M be the submatrix of $R(G, \mathbf{p})$ indexed by F and X. For each $i \in \{1, \ldots, \mathtt{D}\}$, the sum of the ith columns of M indexed by each $v \in X$ is $\mathbf{0}$. Therefore, $|F| = \operatorname{rank} M \leq \mathtt{D}|X| - \mathtt{D}$. Now, Theorem 5.1.1 of Nash-Williams implies that H is the edge-disjoint union of $\mathtt{D}$ spanning trees.

Conversely, let $F_{h,k}, 0 \leq h < k \leq d$, be $\mathtt{D}$ edge-disjoint spanning trees in G. An infinitesimally rigid d-dimensional realization $(G, \mathbf{p})$ of G can be constructed as follows. For an edge $e = uv \in E$, if $e \in F_{h,k}$ then let $p_u^e = \mathbf{e}^h$, $p_v^e = \mathbf{e}^k$. To an edge $e \in E$ that does not belong to any trees $F_{h,k}$ we can assign an arbitrary pair of points in $\mathbb{R}^d$.

We prove that this realization of G is infinitesimally rigid by showing that every infinitesimal motion of $(G, \mathbf{p})$ is trivial. Indeed, let $\mathbf{q} = (\ldots, q_u, \ldots)$ be an infinitesimal motion of $(G, \mathbf{p})$, then we must have $\langle q_u - q_v\,, T_{h,k} \rangle = 0$ for every $e = uv \in F_{h,k}$, for all $0 \leq h < k \leq d$. Since $F_{h,k}$ are spanning trees of G, for every pair of vertices $u, v \in V$, there is a path $u = v_1, e_1, \ldots, v_{s-1}, e_s, v_s = v$ from u to v in $F_{h,k}$. On this path we have $\langle q_{v_i} - q_{v_{i+1}}\,, T_{h,k} \rangle = 0$. Therefore we deduce that

$$\langle q_u - q_v\,, T_{h,k} \rangle \quad \text{for all } u, v \in V \text{ and } 0 \leq h < k \leq d.$$

However, the family of $\mathtt{D}$ vectors $\{T_{h,k} : 0 \leq h < k \leq d\} \subseteq \mathbb{R}^{\mathtt{D}}$ is a base of $\mathbb{R}^{\mathtt{D}}$ by Lemma 6.2.1. Therefore $q_u - q_v = 0$ for all $u, v \in V$. The theorem follows.

□

6.2.2 Body-bar frameworks with bar-boundary

In this model we imagine that some bodies are linked to an external fixed environment by bars. A body-bar framework with bar-boundary is a realization of a graph

G which may contain loops where non-loop edges are realized as normal bars in body-bar model and loops are realized as bars that link the incident bodies to the external fixed environment. More precisely, a realization of a graph $G = (V, E)$ as a body-bar framework with bar-boundary is a map $\mathbf{p}$ that associates each non-loop edge $e = uv \in E$ with a join $T_e = T(p_u^e, p_v^e)$ where $p_u^e, p_v^e \in \mathbb{R}^d$, and each loop $c \in E$ with a join $T_c = T(p_1^c, p_2^c)$ of two points p_1^c, p_2^c in $\mathbb{R}^d$.

An *infinitesimal motion* of a body-bar framework with bar-boundary $(G, \mathbf{p})$ is a map $\mathbf{q}$ that assigns to each body $u \in V$ a vector $q_u \in \mathbb{R}^{\mathtt{D}}$ such that

- for each non-loop edge $e = uv \in E$,

$$\langle T_e \, , q_u - q_v \rangle = 0,$$

- for each loop $c \in E$ at a vertex u,

$$\langle T_c \, , q_u \rangle = 0.$$

  note that this equation for loops is obtained from the equation for non-loop edge by setting $q_v = 0$, reflecting the fact that the external environment is fixed.

A body-bar framework with bar-boundary $(G, \mathbf{p})$ is said to be *infinitesimally rigid* if the only infinitesimal motion is $q_u = 0$ for all $u \in V$, namely, it is fixed to the external environment.

The *rigidity matrix* for a body-bar framework with bar-boundary is defined in the same way as that for a body-bar framework, but for a loop c at a vertex u, there is only one non-zero block $R_{c,u}$.

It is easy to see that a body-bar framework with bar-boundary is infinitesimally rigid if and only if $\operatorname{Ker} R(G, \mathbf{p}) = 0$, or equivalently, $\operatorname{rank} R(G, \mathbf{p}) = \mathtt{D}|V|$.

A generic realization of G in $\mathbb{R}^d$ as a body-bar framework with bar-boundary is a d-dimensional realization $(G, \mathbf{p})$ such that every submatrix of the rigidity matrix of $R(G, \mathbf{p})$ attains its maximum possible rank for a realization in $\mathbb{R}^d$. It is a common fact that almost all realization of G in $\mathbb{R}^d$ are generic. Moreover, if G has an infinitesimally rigid realization in $\mathbb{R}^d$ then all its generic realization in $\mathbb{R}^d$ are infinitesimally rigid.

The following result characterizes graphs that can be realized as infinitesimally rigid d-dimensional body-bar frameworks with bar-boundary.

Theorem 6.2.3. *Let $G = (V, E)$ be a graph. Consider the 2-grading $E_1 = E$ and $E_2 =$ the set of non-loop edges in E. Then G can be realized as an infinitesimally* *rigid body-bar with bar-boundary if and only if it contains a* $(\mathtt{D},(0,\mathtt{D}))$*-graded tight spanning subgraph.*

Before proceeding to the proof, let us mention a similar result of Katoh and Tanigawa [73] on body-bar frameworks with pre-determined bar-boundary. In that model, the configuration for loops, i.e., bars that link the framework to the external environment, is given and one wants to know whether there exists a configuration for non-loop edges, i.e., internal bars, such that the framework is infinitesimally rigid. Using their result on packing of matroid-based rooted trees (which can be implied from our result in Section 5.4), Katoh and Tanigawa shows the following result. Let $\mathcal{T}(\mathbb{R}^d)$ denote the set $T(p,p')$ for all $p,p'\in\mathbb{R}^d$.

Theorem 6.2.4 ([73])**.** *Let* $G=(V,F\cup R)$ *a graph where* R *is the set of loops of* G *and* F *is the set of non-loop edges. Let* $\mathbf{p}^\circ : R\to\mathcal{T}(\mathbb{R}^d)$ *a pre-determined configuration for the loops in* R*. Then there exists a configuration* $\mathbf{p}: E\to\mathcal{T}(\mathbb{R}^d)$ *such that* $(G,\mathbf{p},\mathbf{p}^\circ)$ *is minimally infinitesimally rigid if and only if*

1. $\{\mathbf{p}^\circ_r : r\in R_v\}$ *is linearly independent,*
2. $|F'|+|R_{F'}|\leq \mathtt{D}|V(F')|-\mathtt{D}+\dim(\{\mathbf{p}^\circ_r : r\in R_{F'}\})$*, for every* $F'\subseteq F$*,*
3. $|F|+|R|=\mathtt{D}|V|$*,*

where R_v *denotes the set of loops at* v *and* $R_{F'}$ *denotes the set of loops at vertices in* $V(F')$*.*

When the pre-determined configuration $\mathbf{p}^\circ$ is chosen to be generic, our Theorem 6.2.3 coincides with Theorem 6.2.4. It is interesting that two different generalizations of Nash-Williams' theorem give the same result in this case.

Proof of Theorem 6.2.3. Suppose that $G=(V,E)$ has a realization $(G,\mathbf{p})$ in $\mathbb{R}^d$ as an infinitesimally rigid body-bar framework with bar-boundary, we show that G has a $(\mathtt{D},(0,\mathtt{D}))$-tight spanning subgraph with the 2-grading defined in the theorem. Since $(G,\mathbf{p})$ is infinitesimally rigid, $\operatorname{rank} R(G,\mathbf{p})=\mathtt{D}|V|$. Therefore, there are $\mathtt{D}|V|$ linearly independent rows in $R(G,\mathbf{p})$. Let H be a spanning subgraph of G whose edge set indexes a maximal linearly independent set of rows of $R(G,\mathbf{p})$. We show that H is a $(\mathtt{D},(0,\mathtt{D}))$-tight spanning subgraph of G. First, note that $|E(H)|=\mathtt{D}|X|$ by the way we choose H. It will suffice to show that H is $(\mathtt{D},(0,\mathtt{D}))$-sparse. Let $X\subseteq V$ and F be the edge set of H induced by X. Let M be the submatrix of $R(G,\mathbf{p})$ indexed by F and X. Then $|F|=\operatorname{rank} M\leq \mathtt{D}|V|$. Now let N be the submatrix of $R(G, \mathbf{p})$ indexed by $F \cap E_2$ and X. Then for each $i \in \{1, \ldots, \mathtt{D}\}$ the sum of the ith column of N indexed by each $v \in X$ is $\mathbf{0}$. Therefore, $|i_2(X)| = |F \cap E_2| = \operatorname{rank} N \leq \mathtt{D}|X| - \mathtt{D}$.

Conversely, suppose that G contains a $(\mathtt{D}, (0, \mathtt{D}))$-tight spanning subgraph H. Then by Theorem 5.2.3, H can be decomposed into $\mathtt{D}$ spanning tight pseudoforests $F_{h,k}$, $0 \leq h < k \leq d$ such that in each tight pseudoforest $F_{h,k}$ there is no cycle containing only non-loop edges. It means that each connected component of $F_{h,k}$ contains exactly one loop. We construct a d-dimensional realization $\mathbf{p}$ of G as follows. For $e \in E(H)$, if $e \in F_{h,k}$ for some $0 \leq h < k \leq d$, then set $T_e = T_{h,k}$. For $e \in E(G) \setminus E(H)$ we assign to e an arbitrary $T_e \in \mathcal{T}(\mathbb{R}^d)$.

We prove that this realization of G is infinitesimally rigid by showing that every infinitesimal motion of $(G, \mathbf{p})$ is trivial. Let $\mathbf{q} = (\ldots, q_u, \ldots)$ be an infinitesimal motion of $(G, \mathbf{p})$. In each connected component of $F_{h,k}$ there is a unique loop at some vertex x. For this loop we have $\langle q_x \,, T_{h,k} \rangle = 0$. On the other hand, using the same argument as that in Theorem 6.2.2 we obtain that, for every u, v in the same connected component of $F_{h,k}$ as x, the equality $\langle q_u - q_v \,, T_{h,k} \rangle = 0$ holds. Therefore, we deduce that $\langle q_u \,, T_{h,k} \rangle = 0$ for every vertex u since $F_{h,k}$ is spanning and each of its connected components contains a loop. Hence, we have proved that $\langle q_u \,, T_{h,k} \rangle = 0$ for every vertex $u \in V$ and every tight pseudoforest $F_{h,k}$. By Lemma 6.2.1, the family of $\mathtt{D}$ vectors $\{T_{h,k}, 0 \leq h < k \leq d\} \subseteq \mathbb{R}^{\mathtt{D}}$ is a base of $\mathbb{R}^{\mathtt{D}}$. Therefore, $q_u = 0$ for every $u \in V$. The theorem follows. $\square$

Remark: The above proof of Theorem 6.2.3 illustrates an application of our decomposition of graded sparse graphs. Another way (suggested by Walter Whiteley) to view a body-bar framework with bar-boundary is to consider the external environment as an additional body and apply the body-bar analysis.

6.3 Body-length-direction frameworks

6.3.1 Introduction

Let $G = (V; L, D)$ be a graph where (L, D) is a bipartition of the edge set of G. We refer to edges in L and D as *length edges* and *direction edges*, respectively.

Let $d \geq 2$ be an integer and $\varphi : D \to \{1, 2, \ldots, d-1\}$. (G, φ) is called a *weighted mixed graph*. A *realization* of (G, φ) is a map $\mathbf{p}$ which associates a pair of points $p_u^e, p_v^e \in \mathbb{R}^d$ with each $e = uv \in L$, and a triple (p_u^e, p_v^e, U_e) with each $e = uv \in D$ where $p_u^e, p_v^e \in \mathbb{R}^d$ and U_e is an $\varphi(e)$-dimensional subspace of $\mathbb{R}^d$ which is orthogonal to $p_u^e - p_v^e$. We imagine that the realization $(G, \varphi, \mathbf{p})$ is equipped with a set of d-dimensional bodies B_u, $u \in V$, and that $p_u^e \in B_u$ for all edges e incident to u.

An *infinitesimal motion* of $(G, \varphi, \mathbf{p})$ is a map $\mathbf{q}$ which assigns an instantaneous velocity to each body B_u in such a way that $(q_u^e - q_v^e)$ is orthogonal to $(p_u^e - p_v^e)$ for each $e = uv \in L$ and $(q_u^e - q_v^e)$ is orthogonal to U_e for each $e = uv \in D$, where q_u^e is the instantaneous velocity of p_u^e induced by q_u. The realization $(G, \varphi, \mathbf{p})$ is *infinitesimally rigid* if the only infinitesimal motions of $(G, \varphi, \mathbf{p})$ are translations. We will consider three different types of bodies which allow three different kinds of instantaneous velocities for each body.

1. *Length-direction-rigid bodies* which keep the distances and directions between all pairs of points fixed. These allow only translations as instantaneous velocities.

2. *Direction-rigid bodies* which keep the directions between all pairs of points fixed. These allow only translations and dilations as instantaneous velocities.

3. *Length-rigid bodies* which keep the distances between all pairs of points fixed. These allow only translations and rotations as instantaneous velocities.

Let $G = (V; L, D)$ and $\varphi : D \to \{1, 2, \ldots, d-1\}$. We consider the graph $G^\varphi = (V, L \cup D^\varphi)$ obtained from G by replacing each direction edge $e \in D$ with $\varphi(e)$ copies of itself. We call this graph the *augmented graph of G by φ*. We will also refer to edges in L and D^φ of G^φ as length edges and direction edges, respectively.

Contribution: We give a complete characterization of infinitesimal rigidity of generic body-length-direction frameworks with length-direction-rigid bodies and those with direction-rigid bodies (Theorems 6.3.2 and 6.3.3). We obtain a necessary condition for the infinitesimal rigidity of generic body-length-direction frameworks and show that it is also sufficient in the special case where $\varphi(e) = 1$ for all $e \in D$.

Results in this section are from a joint work with Bill Jackson [66].

We will need the following elementary result concerning the rank of a modified edge/vertex incidence matrix for a pseudoforest. We consider the entries as elements of $\mathbb{F}_2$, the field of integers modulo two, and denote this rank by rank_2.

Lemma 6.3.1. *Let $P = (V, E)$ be a pseudoforest and $F \subseteq E$ such that each cycle in P contains an edge of F. Choose an orientation for P such that each cycle is a directed cycle. Let M be the $|E| \times |V|$ matrix with rows indexed by E and columns indexed by V, in which:*

- *the entries in the row indexed by an edge $e = uv$ oriented from u to v, and the vertices u and v are $1, 0$ if $e \in F$, and $1, 1$ if $e \notin F$;*
- *all other entries are zero.*

Then $\text{rank}_2 M = |E|$.

Proof. If P is not connected then we may apply the lemma inductively to each connected component of P. Similarly, if P has a vertex of v of degree one, then we may apply the lemma inductively to $P - v$. Hence we may assume that P is a cycle, say $P = v_1 e_1 v_2 e_2 \dots e_{n-1} v_n e_n v_1$, and we may suppose further that P is oriented in this cyclic order. Since P has at least one edge in F, we may also suppose that $e_n \in F$. This makes M an upper triangular matrix with ones down the diagonal and the lemma follows. $\square$

6.3.2 Frameworks with length-direction-rigid bodies

We consider realizations of (G, φ) as a d-dimensional body-length-direction framework with length-direction-rigid bodies. We may assume that G has no loops. Let $(G, \varphi, \mathbf{p})$ be such a realization. An infinitesimal motion $\mathbf{q}$ of $(G, \varphi, \mathbf{p})$ can be described by assigning to each body $u \in V$, a vector $q_u \in \mathbb{R}^d$ such that $(q_u - q_v)$ is orthogonal to $(p_u^e - p_v^e)$ for each $e = uv \in L$ and $(q_u - q_v)$ is orthogonal to U_e for each $e = uv \in D$.

It is convenient to think of $\mathbf{p}$ as assigning a vector $m^f \in \mathbb{R}^d$ to each edge f of G^φ such that for each $e \in D$, the set of all vectors m^{e_i} assigned to the $\varphi(e)$ copies of e are linearly independent. We can then choose $p_u^e, p_v^e \in \mathbb{R}^d$ for each $e \in D \cup L$ such that $m^e = p_u^e - p_v^e$ for each $e = uv \in L$, and such that $p_u^e - p_v^e$ is orthogonal to $U_e = \langle m^{e_i} : 1 \leq i \leq \varphi(e)\rangle$ for each $e = uv \in D$, without affecting the infinitesimal rigidity of $(G, \varphi, \mathbf{p})$.

The rigidity matrix $R(G, \varphi, \mathbf{p})$ of $(G, \varphi, \mathbf{p})$ is the $(|L| + |D^\varphi|) \times d|V|$ matrix where each edge of G^φ indexes one row and each vertex indexes d columns so that

- the submatrices indexed by an edge $f = uv$ and the vertices u and v are m^f and $-m^f$, respectively;
- all other entries are 0's.

Then $\mathbf{q}$ is an infinitesimal motion of $(G, \varphi, \mathbf{p})$ if and only if $\mathbf{q} \in \operatorname{Ker} R(G, \varphi, \mathbf{p})$. Thus $(G, \varphi, \mathbf{p})$ is infinitesimally rigid if and only if all infinitesimal motions $\mathbf{q}$ satisfy $q_u = q_v$ for all $u, v \in V$ or, equivalently, $\operatorname{rank} R(G, \varphi, \mathbf{p}) = d|V| - d$.

Theorem 6.3.2. *A weighted mixed graph (G, φ) has a realization as an infinitesimally rigid d-dimensional body-length-direction framework with length-direction-rigid bodies if and only if G^φ has a (d, d)-tight spanning subgraph.*

Proof. Suppose that (G, φ) has an infinitesimally rigid realization $(G, \varphi, \mathbf{p})$ in $\mathbb{R}^d$. Then $\operatorname{rank} R(G, \varphi, \mathbf{p}) = d|V| - d$. Hence $R(G, \varphi, \mathbf{p})$ contains $d|V| - d$ independent rows. Let H be the spanning subgraph of G^φ whose edge set indexes these rows. We show that H is (d, d)-tight. Since $|E(H)| = \operatorname{rank} R(G, \varphi, \mathbf{p}) = d|V| - d$, it is sufficient to show that H is (d, d)-sparse. Let $X \subseteq V$ and let F be the set of edges of H induced by X. Let M be the submatrix of $R(G, \varphi, \mathbf{p})$ indexed by F and X. For each $i \in \{1, 2, \ldots, d\}$, the definition of $R(G, \varphi, \mathbf{p})$ implies that the sum of the ith columns of M indexed by each $v \in X$ is $\mathbf{0}$. Hence $|F| = \operatorname{rank} M \leq d|X| - d$. This implies that H is (d, d)-sparse.

For sufficiency, we will use the decomposition of a (d, d)-tight spanning subgraph of G^φ into spanning trees to construct an infinitesimally rigid realization for (G, φ). Let H be a (d, d)-tight spanning subgraph of G^φ. By Theorem 5.1.1, H is the edge-disjoint union of d spanning trees $T_1, \ldots, T_d$. Let $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^d$ denote the canonical base of $\mathbb{R}^d$, i.e., $\mathbf{e}^i$ is the d-dimensional vector whose ith entry is 1 and all other entries are 0's. For $i = 1, \ldots, d$, and edges $f \in E(T_i)$, we set $m^f = \mathbf{e}^i$. To each $f \notin E(T_1) \cup \cdots \cup E(T_d)$ we assign an $m^f \in \mathbb{R}^d$ such that, for each $e \in D$, the set $\{m^{e_i} : e_i \text{ is a copy of } e \text{ in } G^\varphi\}$ is linearly independent. Recall that each edge of

G^φ indexes one row of of $R(G,\varphi,\mathbf{p})$ and each vertex of G^φ indexes d columns of $R(G,\varphi,\mathbf{p})$. If we group together the rows corresponding to the edges in each tree T_i and the jth columns corresponding to each vertex $v \in V$ then $R(G,\varphi,\mathbf{p})$ takes the form

$$\begin{array}{c c}
 & \begin{array}{cccc} V^1 & V^2 & & V^d \end{array} \\
\begin{array}{c} T_1 \\ T_2 \\ \vdots \\ T_d \\ \end{array} &
\begin{pmatrix}
A_1 & & & \\
 & A_2 & O & \\
 & O & \ddots & \\
 & & & A_d \\
* & * & * & *
\end{pmatrix}
\end{array}$$

where O's are zero matrices and A_i is the incident matrix of the spanning tree T_i, $i = 1, \ldots, d$. The rank of each A_i is then $|V| - 1$ and hence

$$\operatorname{rank} R(G,\varphi,\mathbf{p}) \geq \sum_{i=1}^{d} \operatorname{rank} A_i = d|V| - d.$$

Therefore $(G,\varphi,\mathbf{p})$ is infinitesimally rigid. The theorem now follows. □

6.3.3 Frameworks with direction-rigid bodies

We consider realizations of (G,φ) as frameworks with *direction-rigid bodies*. We can assume that no direction edge in G is a loop (but length edge loops may exist).

An *infinitesimal motion* of a d-dimensional realization $(G,\varphi,\mathbf{p})$ can be described by assigning, to each body $u \in V$, a pair (t_u, r_u) where t_u is a vector in $\mathbb{R}^d$ (which corresponds to a translation) and r_u is a scalar (which corresponds to a dilation) such that:

- for each non-loop length edge $e = uv \in L$,

$$\langle (t_u + r_u p_u^e) - (t_v + r_v p_v^e), p_u^e - p_v^e \rangle = 0, \tag{6.3.1}$$

- for each length edge loop $e = uu \in L$,

$$r_u = 0, \tag{6.3.2}$$

- for each direction edge $e = uv \in D$,

$$\langle (t_u + r_u p_u^e) - (t_v + r_v p_v^e), m \rangle = 0, \tag{6.3.3}$$

  for every vector $m \in U_e$.

Equation (6.3.1) can be rewritten as

$$\langle t_u - t_v\,, p_u^e - p_v^e\rangle + r_u\langle p_u^e\,, p_u^e - p_v^e\rangle + r_v\langle p_v^e\,, p_v^e - p_u^e\rangle = 0. \tag{6.3.4}$$

Let $p^e = (p_u^e + p_v^e)/2$. Since $p_u^e - p_v^e$ is orthogonal to U_e, equation (6.3.3) can be rewritten as

$$\langle t_u - t_v\,, m\rangle + r_u\langle p^e\,, m\rangle - r_v\langle p^e\,, m\rangle = 0 \tag{6.3.5}$$

for all $m \in U_e$.

Therefore, for the purpose of studying infinitesimal rigidity, a realization $(G, \varphi, \mathbf{p})$ can be described by assigning, to each non-loop edge $e = uv \in L$, a triple $(p_u^e, p_v^e, m^e) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ such that m^e is multiple of $p_u^e - p_v^e$, and, to each edge $e = uv \in D$, a point $p^e \in \mathbb{R}^d$ and $\varphi(e)$ linearly independent vectors $m_1^e, \ldots, m_{\varphi(e)}^e \in \mathbb{R}^d$.

The rigidity matrix $R(G, \varphi, \mathbf{p})$ of $(G, \varphi, \mathbf{p})$ is the $(|L|+|D^\varphi|)\times(d+1)|V|$ matrix where each length edge indexes one row, each direction edge indexes $\varphi(e)$ rows and each vertex indexes $d+1$ columns so that:

- the matrices indexed by a non-loop edge $e = uv \in L$ and the vertices u and v are $R_{e,u} = (m^e, \langle p_u^e\,, m^e\rangle)$ and $R_{e,v} = (-m^e, -\langle p_v^e\,, m^e\rangle)$, respectively;
- the matrix indexed by a loop edge $e \in L$ at a vertex v is $R_{e,v} = (0, \ldots, 0, 1)$;
- the matrices indexed by an edge $e = uv \in D$ and the vertices u and v are $R_{e,u}$ and $R_{e,v} = -R_{e,u}$, respectively, where

$$R_{e,u} = \begin{pmatrix} m_1^e & \langle p^e\,, m_1^e\rangle \\ \vdots & \vdots \\ m_{\varphi(e)}^e & \langle p^e\,, m_{\varphi(e)}^e\rangle \end{pmatrix};$$

- elsewhere all the entries are 0's.

Then $\operatorname{Ker} R(G, \varphi, \mathbf{p})$ is the space of all the infinitesimal motions of $(G, \varphi, \mathbf{p})$ and therefore $(G, \varphi, \mathbf{p})$ is rigid if and only if $\operatorname{rank} R(G, \varphi, \mathbf{p}) = (d+1)|V| - d$.

Theorem 6.3.3. *Let $G = (V; L, D)$ be a mixed graph and $\varphi : D \to \{1, \ldots, d-1\}$. Define a 2-grading (E_1, E_2) for the augmented graph G^φ by letting E_1 be the set of all edges of G^φ and E_2 be the set of all direction edges in G^φ. Then (G, φ) has a d-dimensional realization as an infinitesimally rigid body-length-direction framework with direction-rigid bodies if and only if G^φ has a $(d+1; (d, d+1))$-graded-tight spanning subgraph.*

Proof. First suppose that $(G, \varphi, \mathbf{p})$ is an infinitesimally rigid realization of (G, φ) in $\mathbb{R}^d$. Let H be a spanning subgraph of G^φ whose edge set indexes a maximal linearly independent set of rows in the rigidity matrix $R(G, \varphi, \mathbf{p})$. We show that H is a $(d+1; (d, d+1))$-graded-tight spanning subgraph of G^φ. Since $|E(H)| = \operatorname{rank} R(G, \varphi, \mathbf{p}) = (d+1)|V| - d$, it is sufficient to show that H is $(d+1; (d, d+1))$-graded sparse. Let $X \subset V$ and let F be the set of edges of H induced by X. Let M be the submatrix of $R(G, \varphi, \mathbf{p})$ indexed by F and X. For each $i \in \{1, 2, \ldots, d\}$, the definition of $R(G, \varphi, \mathbf{p})$ implies that the sum of the ith columns of M indexed by each $v \in X$ is 0. Hence $|F| = \operatorname{rank} M \leq d|X| - d$. Similarly, if N is the submatrix of $R(G, \varphi, \mathbf{p})$ indexed by $F \cap E_2$ and X, then the sum of the ith columns of N indexed by each $v \in X$ is 0 for all $i \in \{1, 2, \ldots, d+1\}$. Hence $|F \cap E_2| = \operatorname{rank} N \leq d|X| - d - 1$. This implies that H is $(d+1; (d, d+1))$-graded sparse.

For sufficiency, we will use the decomposition of a $(d+1; (d, d+1))$-graded-tight spanning subgraph of G^φ into spanning pseudoforests to construct an infinitesimally rigid realization $R(G, \varphi, \mathbf{p})$ for (G, φ). Let H be a $(d+1; (d, d+1))$-graded-tight spanning subgraph of G^φ. By Theorem 5.2.3, H is the edge-disjoint union of d spanning trees $T_1, \ldots, T_d$ and one spanning tight pseudoforest P. Let $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^d$ denote the canonical base of $\mathbb{R}^d$, and $\mathbf{1} = \mathbf{e}^1 + \ldots + \mathbf{e}^d$ be the all one vector in $\mathbb{R}^d$. For $f \in E(H)$, we set $m^f = \mathbf{e}^i$ if $f \in E(T_i)$ for some $1 \leq i \leq d$ and $m^f = \mathbf{1}$ if $f \in E(P)$. To each $f \notin E(H)$ we assign an $m^f \in \mathbb{R}^d$ such that, for each $e \in D$, the set $\{m^{e_i} : e_i \text{ is a copy of } e \text{ in } G^\varphi\}$ is linearly independent. We associate each row of $R(G, \varphi, \mathbf{p})$ with an edge of G^φ in the obvious way. If we group together the rows corresponding to the edges in each pseudoforest and the jth columns corresponding to each vertex $v \in V$ then $R(G, \varphi, \mathbf{p})$ takes the form

$$
\begin{array}{c}
\\ T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_d \\ P \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} V^1 & V^2 & V^3 & & V^d & V^{d+1} \end{array} \\
\begin{pmatrix}
A_1 & O & O & \ldots & O & A'_1 \\
O & A_2 & O & \ldots & O & A'_2 \\
O & O & A_3 & \ldots & O & A'_3 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
O & O & O & \ldots & A_d & A'_d \\
B & B & B & \ldots & B & B' \\
* & * & * & * & * & *
\end{pmatrix}
\end{array}
$$

where O's are zero matrices, A_i is the edge/vertex incident matrix of T_i for $i = 1, \ldots, d$, B is the edge/vertex incident matrix of P, and A'_i and B' are 'modified' edge/vertex incidence matrices for T_i and P, respectively. (Their non-zero entries occur only in positions which contain non-zero entries in the corresponding incidence matrices but the values depend on p^e_u, p^e_v for $e = uv \in L$ and p^e for $e \in D$, which we will choose next.)

For each $e \in D$ with copies $e_1, e_2, \dots, e_{\varphi(e)}$ in G^φ, choose an index $k_e \in \{1, 2, \dots d-1\}$ such that $e_j \notin T_{k_e}$ for all $1 \le j \le \varphi(e)$ and put $p^e = \mathbf{e}^{k_e}$. For $e = uv \in L \cap T_i$ let $p^e_u = 2\mathbf{e}^i$ and $p^e_v = \mathbf{0}$. The resulting entries in A'_i are as follows.

- The entries indexed by a direction edge $e_j = uv$ in T_i, which is a copy of an edge $e \in D$, and the vertices u and v are $\langle p^e, m^e\rangle = \langle \mathbf{e}^{k_e}, \mathbf{e}^i\rangle = 0$ and $-\langle p^e, m^e\rangle = 0$, respectively.

- The entries indexed by a length edge $e = uv$ in T_i and the vertices u and v are $\langle p^e_u, m^e\rangle = \langle 2\mathbf{e}^i, \mathbf{e}^i\rangle = 2$ and $-\langle p^e_v, m^e\rangle = -\langle \mathbf{0}, \mathbf{1}\rangle = 0$, respectively.

- Elsewhere all the entries are 0's.

Finally, we specify p^e_u, p^e_v for $e \in L \cap P$. We first choose an orientation of the tight pseudoforest P such that each cycle in P is a directed cycle. For each non-loop length edge $e = uv$ of P which has been oriented from u to v, put $p^e_u = \frac{1}{d}\mathbf{1}$ and $p^e_v = \mathbf{0}$. The resulting entries in B' are as follows.

- The entries indexed by a non-loop length edge $e = uv$ in P which has been oriented from u to v, and the vertices u and v are $\langle p^e_u, m^e\rangle = \langle \frac{1}{d}\mathbf{1}, \mathbf{1}\rangle = 1$ and $-\langle p^e_v, m^e\rangle = -\langle \mathbf{0}, \mathbf{1}\rangle = 0$, respectively.

- The entry indexed by a loop length edge $e = vv$ in P and the vertex v is 1.

- The entries indexed by a direction edge e_1 in P which is a copy of an edge $e = uv \in D$, and the vertices u and v are $\langle p^e, m^e_1\rangle = \langle \mathbf{e}^{k_e}, \mathbf{1}\rangle = 1$ and $-\langle p^e, m^e_1\rangle = -1$, respectively.

- Elsewhere all the entries are 0's.

In particular all the entries of $R(G, \varphi, \mathbf{p})$ are integers and all the entries of A'_i, $1 \le i \le d$, are even integers. Since $A'_i = O$ over $\mathbb{F}_2$, we can now use Lemma 6.3.1 to deduce that

$$\begin{aligned}\operatorname{rank} R(G,\varphi,\mathbf{p}) \ge \operatorname{rank}_2 R(G,\varphi,\mathbf{p}) &\ge \sum_{i=1}^{d} \operatorname{rank}_2 A_i + \operatorname{rank}_2 B' \\ &= d|V| - d + |V| = (d+1)|V| - d.\end{aligned}$$

Hence $(G, \varphi, \mathbf{p})$ is infinitesimally rigid. $\square$

Note that a direction-rigid body becomes length-direction-rigid if we add any length constraint to it. This simple observation allows us to determine whether a generic body-length-direction framework with a mixture of direction-rigid and direction-length-rigid bodies is infinitesimally rigid. We just apply Theorem 6.3.3 to the mixed graph we obtain by adding a length edge loop at the vertices which correspond to direction-length-rigid bodies. In particular, we can use this approach to deduce Theorem 6.3.2 from Theorem 6.3.3.

6.3.4 Frameworks with length-rigid bodies

We consider realizations of (G, φ) as frameworks with *length-rigid bodies*. We may assume that no length edge is a loop in G (but direction edge loops may exist). An *infinitesimal motion* of such a realization $(G, \varphi, \mathbf{p})$ is an assignment, to each body $u \in V$, of a pair (t_u, A_u) where t_u is a vector in $\mathbb{R}^d$ (corresponding to a translation) and A_u is a $d \times d$ skew symmetric matrix (corresponding to a rotation) such that:

- for a length edge $e = uv$,
$$\langle t_u + A_u p_u^e - (t_v + A_v p_v^e)\,, p_u^e - p_v^e \rangle = 0; \tag{6.3.6}$$
- for a non-loop direction edge $e = uv$,
$$\langle t_u + A_u p_u^e - (t_v + A_v p_v^e)\,, m \rangle = 0, \tag{6.3.7}$$
for every vector m in U_e.
- for a loop direction edge $e = uu$,
$$\langle A_u (p_{u,1}^e - p_{u,2}^e)\,, m \rangle = 0, \tag{6.3.8}$$
for all $m \in U_e$.

(See Section 2.6.)

For each $u \in V$, let w_u be the $\binom{d}{2}$-dimensional vector associated with the skew-symmetric matrix A_u defined as in Section 2.5.

Equation (6.3.6) can be rewritten as

$$\langle t_u - t_v\,, p_u^e - p_v^e \rangle + \langle A_u p_u^e\,, p_u^e - p_v^e \rangle - \langle A_v p_v^e\,, p_u^e - p_v^e \rangle = 0,$$

which is then equivalent to

$$\langle t_u - t_v\,, p_u^e - p_v^e \rangle + \langle w_u\,, (p_u^e - p_v^e) \vee p_u^e \rangle - \langle w_v\,, (p_u^e - p_v^e) \vee p_v^e \rangle = 0. \tag{6.3.9}$$

If we put $m^e = p_u^e - p_v^e$ and $p^e = (p_u^e + p_v^e)/2$ then equation (6.3.9) is equivalent to

$$\langle t_u - t_v \, , m^e \rangle + \langle w_u \, , m^e \vee p^e \rangle - \langle w_v \, , m^e \vee p^e \rangle = 0. \tag{6.3.10}$$

Similarly, equations (6.3.7) and (6.3.8) can be rewritten as

$$\langle t_u - t_v \, , m \rangle + \langle w_u \, , m \vee p_u^e \rangle - \langle w_v \, , m \vee p_v^e \rangle = 0 \tag{6.3.11}$$

and

$$\langle w_u \, , m \vee (p_{u,1}^e - p_{u,2}^e) \rangle = 0 \tag{6.3.12}$$

for each $m \in U_e$.

It follows that, for the purpose of studying infinitesimal rigidity, we can think of $\mathbf{p}$ as assigning, to each length edge $e \in L$, a pair $(p^e, m^e) \in \mathbb{R}^d \times \mathbb{R}^d$, and, to each direction edge $e = uv \in D$, a pair $(p_u^e, p_v^e) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\varphi(e)$ independent vectors $m_1, \ldots, m_{\varphi(e)} \in \mathbb{R}^d$ orthogonal to $p_u^e - p_v^e$. The *rigidity matrix* $R(G, \varphi, \mathbf{p})$ of this realization is a $(|L| + |D^\varphi|) \times \binom{d+1}{2}|V|$ matrix where each length edge $e \in L$ indexes one row, each direction edge $e \in D$ indexes $\varphi(e)$ rows and each vertex indexes $\binom{d+1}{2}$ columns such that:

- the submatrices indexed by a length edge $e = uv \in L$ and the vertices u and v are $R^e = (m^e \quad m^e \vee p^e)$ and $-R^e$, respectively;

- the submatrices indexed by a non-loop direction edge $e = uv \in D$ and the vertices u and v are

$$R_u^e = \begin{pmatrix} m_1 & m_1 \vee p_u^e \\ \vdots & \vdots \\ m_{\varphi(e)} & m_{\varphi(e)} \vee p_u^e \end{pmatrix} \quad \text{and} \quad R_v^e = \begin{pmatrix} -m_1 & -m_1 \vee p_v^e \\ \vdots & \vdots \\ -m_{\varphi(e)} & -m_{\varphi(e)} \vee p_v^e \end{pmatrix}$$

  respectively;

- the submatrix indexed by a loop direction edge $e = uu \in D$ and the vertex u is

$$R_u^e = \begin{pmatrix} \mathbf{0} & m_1 \vee (p_{u,1}^e - p_{u,2}^e) \\ \vdots & \vdots \\ \mathbf{0} & m_{\varphi(e)} \vee (p_{u,1}^e - p_{u,2}^e) \end{pmatrix}$$

- elsewhere all entries are 0's.

Then $\operatorname{Ker} R(G, \varphi, \mathbf{p})$ is the space of all the infinitesimal motions of $(G, \varphi, \mathbf{p})$ and therefore $(G, \varphi, \mathbf{p})$ is infinitesimally rigid if and only if $\operatorname{rank} R(G, \varphi, \mathbf{p}) =$

$\binom{d+1}{2}|V| - d$.

The following result gives a necessary condition for a graph (G, φ) to have an infinitesimally rigid realization as a d-dimensional framework with length-rigid bodies.

Theorem 6.3.4. *Let $G = (V; L, D)$ be a mixed graph and $\varphi : D \to \{1, \ldots, d-1\}$. Define a 2-grading (E_1, E_2) for the augmented graph G^φ by letting E_1 be the set of all edges of G^φ and E_2 be the set of all length edges in G^φ.*

(a) *If (G, φ) has a d-dimensional realization as an infinitesimally rigid body-length-direction framework with length-rigid bodies then the augmented graph G^φ has a $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded-tight spanning subgraph.*

(b) *If $\varphi(e) = 1$ for all $e \in D$ and G^φ has a $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded-tight spanning subgraph then (G, φ) has a d-dimensional realization as an infinitesimally rigid body-length-direction framework with length-rigid bodies.*

Proof. (a) Suppose that $(G, \varphi, \mathbf{p})$ is an infinitesimally rigid realization of (G, φ) in $\mathbb{R}^d$. Let H be a spanning subgraph of G^φ whose edge set indexes a maximal linearly independent set of rows in the rigidity matrix $R(G, \varphi, \mathbf{p})$. We show that H is a $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded-tight spanning subgraph of G^φ. Since $|E(H)| = \operatorname{rank} R(G, \varphi, \mathbf{p}) = \binom{d+1}{2}|V| - d$, it is sufficient to show that H is $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded sparse. Let $X \subset V$ and let F be the set of edges of H induced by X. Let M be the submatrix of $R(G, \varphi, \mathbf{p})$ indexed by F and X. For each $i \in \{1, 2, \ldots, d\}$, the definition of $R(G, \varphi, \mathbf{p})$ implies that the sum of the ith columns of M indexed by each $v \in X$ is 0. Hence $|F| = \operatorname{rank} M \le \binom{d+1}{2}|X| - d$. Similarly, if N be the submatrix of $R(G, \varphi, \mathbf{p})$ indexed by $F \cap E_2$ and X, then the sum of the ith columns of N indexed by each $v \in X$ is 0 for all $i \in \{1, 2, \ldots, \binom{d+1}{2}\}$. Hence $|F \cap E_2| = \operatorname{rank} N \le \binom{d+1}{2}|X| - \binom{d+1}{2}$. This implies that H is $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded sparse.

(b) Suppose that $\varphi(e) = 1$ for all $e \in D$ and G^φ has a $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded-tight spanning subgraph H. Then $G = G^\varphi$ and H is a spanning subgraph of G. Hence it will suffice to show that H has a d-dimensional realization as an infinitesimally rigid body-length-direction framework with length-rigid bodies. By Theorem 5.2.3, H is the edge-disjoint union of d spanning trees $T_1, \ldots, T_d$ and $\binom{d}{2}$ spanning tight pseudoforests $P_{i,j}$, $1 \le i < j \le d$. Let $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^d$ denote the canonical base of $\mathbb{R}^d$. Choose an orientation for each tight pseudoforest $P_{i,j}$ such that each cycle in $P_{i,j}$ is a directed cycle. We next choose a realization $\mathbf{p}$ for H as follows.

- For $f = uv \in E(T_i)$, we set $m^f = \mathbf{e}^i$. We also choose $j_f \in \{1, 2, \ldots, d\} \setminus \{i\}$ and put $p^f = 2\mathbf{e}^{j_f}$ if $f \in L$, and $p^f_u = 2\mathbf{e}^{j_f}$ and $p^f_v = 0$ if $f \in D$.
- For $f = uv \in E(P_{i,j})$, we set $m^f = \mathbf{e}^i - \mathbf{e}^j$. We also put $p^f = \frac{1}{2}(\mathbf{e}^i + \mathbf{e}^j)$ if $f \in L$, and $p^f_u = \frac{1}{2}(\mathbf{e}^i + \mathbf{e}^j)$ and $p^f_v = 0$ if $f \in D$ and f is oriented from u to v in $P_{i,j}$ (taking p^f_u, p^f_v to be $p^f_{u,1}, p^f_{u,2}$ respectively, when $f = uu$ is a loop).

We index the columns of $R(H, \mathbf{p})$ corresponding to each vertex $v \in V$ by v^i, $1 \le i \le d$ and $v^{i,j}$, $1 \le i < j \le d$. If we group together the rows corresponding to the edges in each pseudoforest and the columns indexed by the same copy of each $v \in V$ then $R(H, \mathbf{p})$ takes the form

$$\begin{array}{c} \\ T_1 \\ T_2 \\ \vdots \\ T_d \\ P_{1,1} \\ \vdots \\ P_{d-1,d} \end{array}
\begin{array}{c}
\begin{array}{cccccc} V^1 & V^2 & V^3 & & V^d & \end{array} \\
\begin{pmatrix}
A_1 & O & O & \ldots & O & A'_1 \\
O & A_2 & O & \ldots & O & A'_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
O & O & O & \ldots & A_d & A'_d \\
* & * & * & \ldots & * & B_{1,1} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
* & * & * & \ldots & * & B_{d-1,d}
\end{pmatrix}
\end{array}$$

where O's are zero matrices and A_i is the edge/vertex incident matrix of T_i for $i = 1, \ldots, d$. Furthermore, the entries in $R(H, \mathbf{p})$ are all integers, the entries in $A'_1, \ldots, A'_d$ are all even integers, and

$$\begin{pmatrix} B_{1,1} \\ B_{1,2} \\ \vdots \\ B_{d-1,d} \end{pmatrix} = \begin{pmatrix} B'_{1,1} & O & \ldots & O \\ O & B'_{1,1} & \ldots & O \\ \vdots & \vdots & & \vdots \\ O & O & \ldots & B'_{d-1,d} \end{pmatrix}$$

where $B'_{i,j}$ is a modified incidence matrix for $P_{i,j}$. The entries of $B'_{i,j}$ in the row indexed by a non-loop edge $f = uv$ of $P_{i,j}$ and the columns indexed by the vertices u, v are: $1, -1$ if $f \in L$; $1, 0$ if $f \in D$ and is oriented from u to v. The entry in the row indexed by a loop direction edge $f = uu$ and the column indexed by u is 1. All other entries are zero. We can now use Lemma 6.3.1 to deduce as in the proof of Theorem 6.3.3 that

$$\operatorname{rank} R(H, \mathbf{p}) \ge \operatorname{rank}_2 R(H, \mathbf{p}) \ge d(|V| - 1) + \binom{d}{2}|V| = \binom{d+1}{2}|V| - d$$

and hence $(H, \mathbf{p})$ is infinitesimally rigid. $\square$

We conjecture that the hypothesis "$\varphi(e) = 1$ for all $e \in D$" is not needed in (b), and hence that the condition "G^φ has a $(\binom{d+1}{2}, (d, \binom{d+1}{2}))$-graded-tight spanning subgraph" is both necessary and sufficient for (G, φ) to have a d-dimensional infinitesimally rigid realization for all φ.

6.4 Direction-length frameworks

6.4.1 Introduction

A *direction-length graph* is a multigraph $G = (V, D \cup L)$ without loops in which the edge set is partitioned into two sets D and L such that neither D nor L contains parallel edges. Edges in D are called *direction edges* and edges in L are called *length edges.* A d-dimensional *direction-length framework* is a pair $(G, \mathbf{p})$ of a direction-length graph G and a map $\mathbf{p}$ which maps each $v \in V$ to a point $p(v)$ in $\mathbb{R}^d$. We can also regard $\mathbf{p}$ as a (row) vector in $\mathbb{R}^{nd}$, where $n = |V|$ is the number of vertex of G. We say that $(G, \mathbf{p})$ is a *direction-length realization* of G in $\mathbb{R}^d$. To illustrate direction-length frameworks we will use dashed segments to denote direction edges and continuous segments to denote length edges.

Two realizations $(G, \mathbf{p})$ and $(G, \mathbf{q})$ of G in $\mathbb{R}^d$ are *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for every length edge uv and there exists a scalar λ_{uv} such that $p(u) - p(v) = \lambda_{uv}(q(u) - q(v))$ for every direction edge uv. They are *congruent* if there exists a $\lambda \in \{1, -1\}$ such that $p(u) - p(v) = \lambda(q(u) - q(v))$ for every u, v in V, i.e. $(G, \mathbf{q})$ can be obtained from $(G, \mathbf{p})$ by a translation and, possibly, a dilation by -1.

A direction-length framework $(G, \mathbf{p})$ is *locally rigid* if there exists an $\epsilon > 0$ such that, for every equivalent realization $(G, \mathbf{q})$ of $(G, \mathbf{p})$, if $\|\mathbf{p} - \mathbf{q}\| < \epsilon$ then $(G, \mathbf{q})$ is congruent to $(G, \mathbf{p})$. Equivalently, every continuous motion of $(G, \mathbf{p})$ respecting direction and length constraints results in a framework that is congruent to $(G, \mathbf{p})$.

An *infinitesimal motion* of a direction-length framework $(G, \mathbf{p})$ is an assignment to each vertex $v \in V$ a vector $\mu(v) \in \mathbb{R}^d$ such that

$$\begin{aligned}
(\mu(u) - \mu(v))(p(u) - p(v)) = 0 &\quad \text{for each length edge } uv \in L, \\
\mu(u) - \mu(v) \text{ is parallel to } p(u) - p(v) &\quad \text{for each direction edge } uv \in D.
\end{aligned}$$

For ease of notation, let $[p(u) - p(v)]^\perp$ denote a $(d-1) \times d$ matrix whose rows form a base of the subspace of $\mathbb{R}^d$ orthogonal to $p(u) - p(v)$. We may suppose throughout the sequel that the rows of this matrix are orthonormal.

The *rigidity matrix* $R(G, \mathbf{p})$ is defined as follows. $R(G, \mathbf{p})$ is a $(|L| + (d-1)|D|) \times d|V(G)|$ matrix. Each vertex of G indexes d columns of $R(G, \mathbf{p})$, each length edge $xy \in L$ indexes one row in $R(G, \mathbf{p})$ and each direction edge $uv \in D$ indexes $(d-1)$ rows of $R(G, \mathbf{p})$ in such a way that,

- the submatrix of $R(G, \mathbf{p})$ indexed by an edge $xy \in L$ and the vertices x, y

are $p(x) - p(y)$ and $p(y) - p(x)$ respectively;

- the submatrix of $R(G, \mathbf{p})$ indexed by an edge $uv \in D$ and the vertices u, v are $[p(u) - p(v)]^\perp$ and $-[p(u) - p(v)]^\perp$ respectively;
- elsewhere all entries are 0's.

Here $p(x) - p(y)$, $p(u) - p(v)$ are regarded as row vectors. That is

$$R(G, \mathbf{p}) = \begin{array}{c} \\ \\ \scriptstyle xy \in L \\ \scriptstyle uv \in D \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \scriptstyle x & \scriptstyle y & \scriptstyle u & \scriptstyle v \end{array} \\ \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ p(x) - p(y) & p(y) - p(x) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [p(u) - p(v)]^\perp & -[p(u) - p(v)]^\perp \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{array}.$$

Then it is easy to see that μ is an infinitesimal motion of $(G, \mathbf{p})$ if and only if $R(G, \mathbf{p})\mu = 0$, where μ is regarded as an nd-dimensional column vector.

A realization $(G, \mathbf{p})$ in $\mathbb{R}^d$ is said to be *(linearly) generic* if the rank of every submatrix of $R(G, \mathbf{p})$ is the maximum among all realizations of G in $\mathbb{R}^d$.

The relation between the rank of the rigidity matrix and the local rigidity of a generic framework is revealed in the following theorem.

Theorem 6.4.1 (Jackson and Keevash [64]). *Let G be a direction-length graph.*

1. *For every embedding $\mathbf{q}$ of G into $\mathbb{R}^d$, $\operatorname{rank} R(G, \mathbf{q}) \le d|V(G)| - d$. Moreover, if $\operatorname{rank} R(G, \mathbf{q}) = d|V(G)| - d$ then for every generic embedding $\mathbf{p}$, $\operatorname{rank} R(G, \mathbf{p}) = d|V(G)| - d$.*
2. *For every generic embedding $\mathbf{p}$, $(G, \mathbf{p})$ is locally rigid if and only if $\operatorname{rank} R(G, \mathbf{p}) = d|V(G)| - d$.*

This theorem implies that local rigidity is a generic property of direction-length frameworks. We say that a direction-length graph G is *generically locally rigid* in $\mathbb{R}^d$ if some (and therefore every) linearly generic realization of G in $\mathbb{R}^d$ as direction-length framework is locally rigid.

Servatius and Whiteley [94] characterized generically locally rigid direction-length graph in $\mathbb{R}^2$ in terms of sparsity.

Theorem 6.4.2 (Servatius and Whiteley [94]). *A direction-length graph $G = (V, D \cup L)$ is generically locally rigid in $\mathbb{R}^2$ if and only if there is a subset F of edges of G satisfying the following conditions.*

1. $|F| = 2|V| - 2$.
2. *For every* $F' \subseteq F$, $|F'| \leq 2|V(F')| - 2$.
3. *For every* $F' \subseteq F$ *such that* F' *contains only length edges or only direction edges,* $|F'| \leq 2|V(F')| - 3$.

The main ingredient in the proof of this characterization is an inductive construction of the direction-length graph (V, F) by 0-extensions and 1-extensions. The 2-dimensional 0-extensions and 1-extensions for direction-length graphs are defined in exactly the same way as those for bar-joint graphs (i.e., length graphs), keeping in mind that the new edges may be direction edges or length edges.

Contribution: We extend the definition of 0-extensions and 1-extensions to d-dimensional case. We show a condition under which these operations preserve the generic local rigidity of direction-length graphs in $\mathbb{R}^d$. Although these operations are not sufficient to characterize generically locally rigid direction-length graphs in general dimension, we obtain a special sufficient condition for a direction-length graph to be generically locally rigid in every dimension, namely, when the direction-length graph contains both a direction spanning tree and a length spanning tree.

6.4.2 0-extensions for direction-length graphs

Definition 6.4.1. *Let* H *be a direction-length graph.*

A length-pure d-dimensional 0-extension *of* H *is a direction-length graph* G *obtained from* H *by adding a new vertex* v *and connecting* v *to* d *distinct vertices* $v_1, \dots, v_d$ *of* H *by length edges (Figure 6.2).*

A d-dimensional direction-pure 0-extension *of* H *is a direction-length graph* G *obtained from* H *by adding a new vertex* v *and connecting* v *to* 2 *distinct vertices* v_1, v_2 *of* H *by direction edges (Figure 6.3).*

A d-dimensional mixed 0-extension *of* H *is a direction-length graph* G *obtained from* H *by adding a new vertex* v *and connecting* v *to* 2 *vertices* v_1, v_2 *of* H *by a length edge and a direction edge (Figure 6.4).*

When there is no need to specify, we use "0-extension" to refer to any kind of 0-extensions defined above. The following result shows that d-dimensional 0-extensions preserve generic local rigidity of direction-length graphs.

Proposition 6.4.3. *Let* H *be a generically locally rigid direction-length graph in* $\mathbb{R}^d$*. Suppose that* G *is a* d*-dimensional* 0*-extension of* H*. Then* G *is generically locally rigid in* $\mathbb{R}^d$*.*

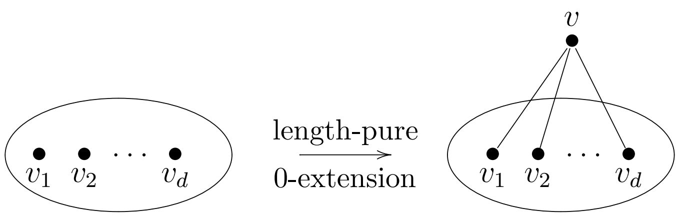


Figure 6.2: Length-pure d-dimensional 0-extension.

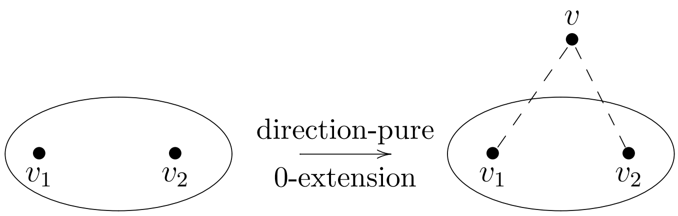


Figure 6.3: Direction-pure d-dimensional 0-extension.

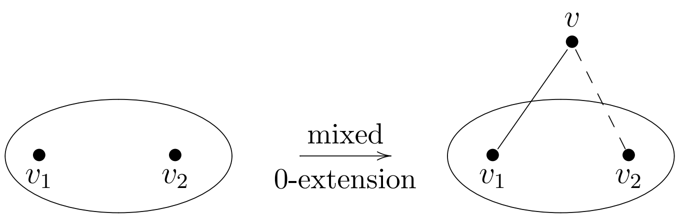


Figure 6.4: Mixed d-dimensional 0-extension.

Proof. Let $\mathbf{p}$ be a generic embedding of G in $\mathbb{R}^d$. By Theorem 6.4.1, it suffices to show that $\operatorname{rank} R(G, \mathbf{p}) \geq d|V(G)| - d$. We consider each possibility.

Case 1: *G is a length-pure d-dimensional 0-extension of H.*

Let $G = H + v + vv_1 + \cdots + vv_d$ where $vv_1, \ldots, vv_d$ are length edges. Then the rigidity matrix of $(G, \mathbf{p})$ has form

$$R(G,\mathbf{p}) = \begin{array}{c} \begin{array}{cc} v & \phantom{R(H,\mathbf{p})} \end{array} \\ \begin{pmatrix} A & * \\ \mathbf{0} & R(H,\mathbf{p}) \end{pmatrix} \end{array},$$

where

$$A = \begin{pmatrix} p(v) - p(v_1) \\ \vdots \\ p(v) - p(v_d) \end{pmatrix}.$$

Since $\mathbf{p}$ is generic, A is non-singular and hence $\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} A + \operatorname{rank} R(H, \mathbf{p}) = d + d|V(H)| - d = d|V(G)| - d$.

Case 2: *G is a direction-pure d-dimensional 0-extension of H.*

Let $G = H + v + vv_1 + vv_2$ where vv_1, vv_2 are direction edges. Then the rigidity matrix of $(G, \mathbf{p})$ has form

$$R(G,\mathbf{p}) = \begin{array}{c} v \\ \begin{pmatrix} B & * \\ \mathbf{0} & R(H,\mathbf{p}) \end{pmatrix} \end{array},$$

where

$$B = \begin{pmatrix} [p(v)-p(v_1)]^{\perp} \\ [p(v)-p(v_2)]^{\perp} \end{pmatrix}.$$

Then since $\mathbf{p}$ is generic, it is easy to see that B is full column rank. Therefore $\operatorname{rank} R(G,\mathbf{p}) \geq \operatorname{rank} B + \operatorname{rank} R(H,\mathbf{p}) = d + d|V(H)| - d = d|V(G)| - d$ holds.

Case 3: *G is a mixed d-dimensional 0-extension of H.*

Let $G = H + v + vv_1 + vv_2$ where vv_1 is a length edge and vv_2 is a direction edge. Let $\mathbf{p}$ be a generic embedding of $V(G)$ in $\mathbb{R}^d$. Then the rigidity matrix of $(G, \mathbf{p})$ has form

$$R(G,\mathbf{p}) = \begin{array}{c} v \\ \begin{pmatrix} C & * \\ \mathbf{0} & R(H,\mathbf{p}) \end{pmatrix} \end{array},$$

where

$$C = \begin{pmatrix} p(v)-p(v_1) \\ [p(v)-p(v_2)]^{\perp} \end{pmatrix}.$$

It also follows from the genericity of $\mathbf{p}$ that C is a non-singular matrix. Therefore $\operatorname{rank} R(G,\mathbf{p}) \geq \operatorname{rank} C + \operatorname{rank} R(H,\mathbf{p}) = d + d|V(H)| - d = d|V(G)| - d$ holds. $\square$

6.4.3 1-extensions for direction-length graphs

We now define the operations of 1-extensions for direction-length graphs in general dimension which extend those for 2-dimensional case introduced by Servatius and Whiteley in [94].

Definition 6.4.2. *Let H be a direction-length graph and v_1v_2 be a length edge in H. A direction-length graph G is said to be a* length-pure d-dimensional 1-extension *of H on edge v_1v_2 if it can be obtained from H by deleting the edge v_1v_2, adding a new vertex v and connecting v to v_1, v_2 and other $d-1$ vertices of H by length edges (Figure 6.5).*

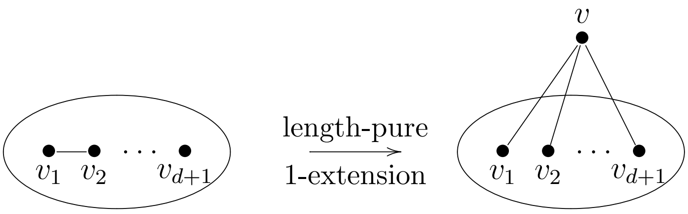


Figure 6.5: Length-pure d-dimensional 1-extension.

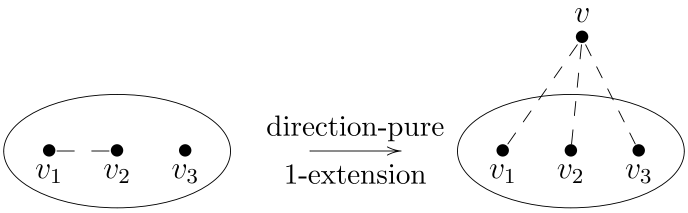


Figure 6.6: Direction-pure d-dimensional 1-extension.

Definition 6.4.3. *Let H be a direction-length graph and v_1v_2 be a direction edge in H. A direction-length graph G is said to be a* direction-pure d-dimensional 1-extension *of H on the edge v_1v_2 if it can be obtained from H by deleting the edge v_1v_2, adding a new vertex v and connecting v to v_1, v_2 and one other vertex of H by direction edges (Figure 6.6).*

Definition 6.4.4. *Let H be a direction-length graph and v_1v_2 be an edge in H (direction edge or length edge). A direction-length graph G is said to be a* reduced mixed d-dimensional 1-extension *of H on the edge v_1v_2 if it can be obtained from H by deleting the edge v_1v_2, adding a new vertex v and connecting v to vertices in H by* 3 *edges e_1, e_2, e_3 such that e_1 is connected to v_1, e_2 is connected to v_2 and there are at least one direction edge and one length edge in $\{e_1, e_2, e_3\}$ (Figure 6.7).*

In fact, one should expect that all the d-dimensional 1-extensions verify the property that

$$\text{the number of new constraints} - \text{the number of deleted constraints} \geq d,$$

noting that each length edge induces one linear constraint while each direction edge induces $d-1$ linear constraints. However, a mixed 1-extension that verifies this property does not necessarily preserve generic local rigidity. To the extent of our results in this section and in the section on global rigidity, the above defined 1-extensions and reduced mixed 1-extensions are convenient and when restricted to dimension 2 they agree with the operations introduced by Servatius and Whiteley

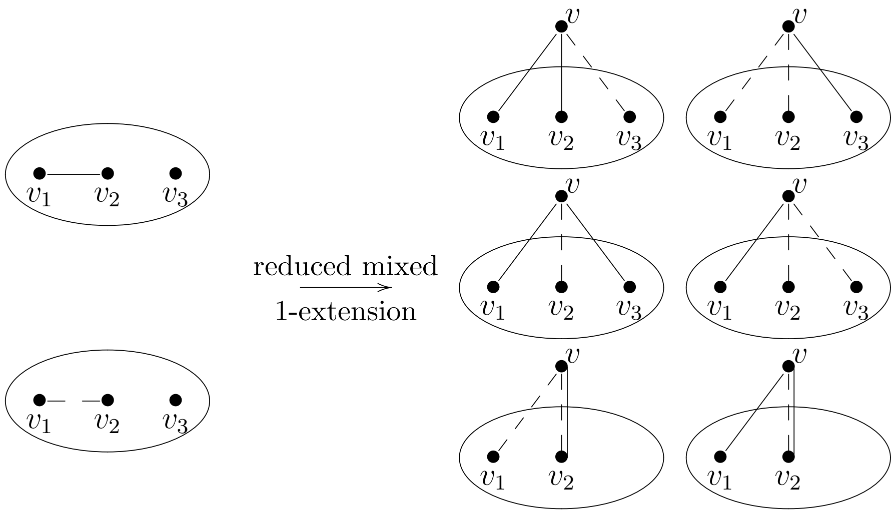


Figure 6.7: Reduced mixed d-dimensional 1-extension.

[94]. When there is no need to specify, we use "1-extension" to refer to any kinds of 1-extensions or reduced 1-extensions defined above.

The next propositions show the generic local rigidity preservingness of 1-extensions under certain conditions.

Proposition 6.4.4. *Let H be a generically locally rigid direction-length graph in $\mathbb{R}^d$ and e be a length edge of H. Suppose that G is a d-dimensional 1-extension of H on e. Then G is generically locally rigid in $\mathbb{R}^d$.*

Proof. Let $e = xy$. We consider the following cases.

Case 1: *G is a length-pure 1-extension of H.*

Let $G = H - xy + vx + vy + vz_1 + \cdots vz_{d-1}$. Let $\mathbf{p}$ be an embeding of G such that the restriction of $\mathbf{p}$ on $V(H)$ is generic, $p(v)$ is the midpoint of the segment $[p(x), p(y)]$, and $p(v)$ is not in a hyperplane containing $p(y), p(z_1), \ldots, p(z_{d-1})$. The three rows of $R(G + xy, \mathbf{p})$ indexed by xy, vx, vy then form a circuit in the linear matroid defined on the rows of this matrix. Therefore,

$$\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} R(G + xy, \mathbf{p}) = \operatorname{rank} R(G + xy - vx, \mathbf{p}).$$

Matrix $R(G + xy - vx, \mathbf{p})$ has form

$$\begin{array}{c} \quad v \qquad\qquad \\ \begin{pmatrix} A & * \\ 0 & R(H, \mathbf{p}) \end{pmatrix}, \end{array}$$

where

$$A = \begin{pmatrix} p(v) - p(y) \\ p(v) - p(z_1) \\ \vdots \\ p(v) - p(z_{d-1}) \end{pmatrix}.$$

Since the restriction of $\mathbf{p}$ on $V(H)$ is generic and $p(v)$ is not in a hyperplane containing $p(y), p(z_1), \ldots, p(z_{d-1})$, A is non-singular. Therefore, $\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} R(G + xy - vx, \mathbf{p}) = \operatorname{rank} A + \operatorname{rank} R(H, \mathbf{p}) = d + d|V(H)| - d = d|V(G)| - d$, since $(H, \mathbf{p})$ is locally rigid. For any generic embedding $\mathbf{q}$ of G, $\operatorname{rank} R(G, \mathbf{q}) \geq \operatorname{rank} R(G, \mathbf{p}) = d|V(G)| - d$ holds. Therefore, G is generically locally rigid by Theorem 6.4.1.

Case 2: *G is a reduced mixed 1-extension of H.*

Let $G = H + vx + vy + vz - xy$.

Case 2.1: *vx, vy are length-edges and vz is a direction edge.*

Taking an embedding $\mathbf{p}$ as in Case 1, we have,

$$\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} R(G + xy, \mathbf{p}) = \operatorname{rank} R(G + xy - vx, \mathbf{p}),$$

and

$$R(G + xy - vx, \mathbf{p}) = \begin{array}{c} \begin{array}{cc} v & \phantom{R(H,\mathbf{p})} \end{array} \\ \begin{pmatrix} B & * \\ 0 & R(H, \mathbf{p}) \end{pmatrix} \end{array},$$

where

$$B = \begin{pmatrix} p(v) - p(y) \\ [p(v) - p(z)]^{\perp} \end{pmatrix}.$$

is a non-singular matrix as the restriction of $\mathbf{p}$ on $V(H)$ is generic and $p(v) = \frac{1}{2}[p(x) + p(y)]$. Then $\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} B + \operatorname{rank} R(H, \mathbf{p}) = d|V(G)| - d$ holds. That implies G is generically locally rigid according to Theorem 6.4.1.

Case 2.2: *vx, vy are direction edges and vz is a length edge.*

Consider an embedding $\mathbf{p}$ of G which is generic on $V(H)$. Let w be a unit vector in $\mathbb{R}^d$ such that w is perpendicular to $p(x) - p(y)$ but not perpendicular to $p(v) - p(z)$. The subspace of $\mathbb{R}^d$ orthogonal to w contains $p(x) - p(y)$, so we can take $w^{\perp}$ with the first row being $(p(x) - p(y))/\|p(x) - p(y)\|$. Let $R^{\infty}(G + xy, \mathbf{p}), R^{\infty}(G + xy - vx, \mathbf{p})$ be the matrices obtained from $R(G + xy, \mathbf{p}), R(G + xy - vx, \mathbf{p})$ respectively by replacing $[p(v) - p(x)]^{\perp}$, $[p(v) - p(y)]^{\perp}$, and $p(v) - p(z)$ with w. Let $R^{\infty}(G, \mathbf{p})$ denote the matrix obtained from $R^{\infty}(G + xy, \mathbf{p})$ by deleting

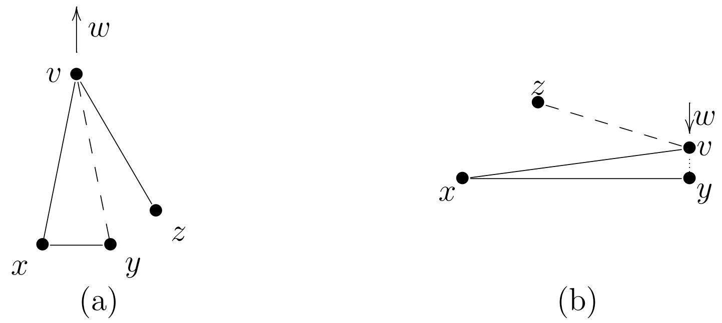


Figure 6.8: (a) v tends to infinity in the direction of w.
(b) v tends to y in the direction of w.

the row indexed by xy. We can view $R^\infty(G, \mathbf{p})$ as the limit of $R(G, \mathbf{p})$ (up to a non-zero scalar scaling of some rows) when $\mathbf{p}$ goes to infinity in the direction of w (Figure 6.8(a)). In the linear matrix defined on the rows of $R^\infty(G + xy, \mathbf{p})$, by the choice of $w^\perp$, the first row of the submatrix indexed by vx, the first row of the submatrix indexed by vy and the row indexed by xy form a circuit. This implies $\operatorname{rank} R^\infty(G, \mathbf{p}) = \operatorname{rank} R^\infty(G + xy, \mathbf{p}) = \operatorname{rank} R^\infty(G + xy - vx, \mathbf{p})$. We have,

$$R^\infty(G + xy - vx, \mathbf{p}) = \begin{array}{c} \begin{array}{cc} v & \phantom{R(H,\mathbf{p})} \end{array} \\ \begin{pmatrix} C & * \\ \mathbf{0} & R(H, \mathbf{p}) \end{pmatrix} \end{array},$$

where

$$C = \begin{pmatrix} w \\ w^\perp \end{pmatrix}$$

is a non-singular matrix. Therefore, $\operatorname{rank} R^\infty(G+xy-vx, \mathbf{p}) \geq \operatorname{rank} R(H, \mathbf{p}) + d = d|V(G)| - d$, since H is generically locally rigid. It follows that $\operatorname{rank} R^\infty(G, \mathbf{p}) \geq d|V(G)| - d$.

Since the rank function of a matrix is lower semi-continuous, there exists an embedding $\mathbf{p}$ of G such that $\operatorname{rank} R(G, \mathbf{p}) \geq \operatorname{rank} R^\infty(G, \mathbf{p}) \geq d|V(G)| - d$. Therefore, G is generically locally rigid by Theorem 6.4.1.

Case 2.3: *vx is a length edge, vy, vz are direction edges.*

Consider $\mathbf{p}, w$ and $w^\perp$ as in Case 2.2. Let $R^\circ(G, \mathbf{p}), R^\circ(G + xy, \mathbf{p})$ and $R^\circ(G + xy - vx)$ be respectively obtained from $R(G, \mathbf{p}), R(G + xy, \mathbf{p}), R(G + xy - vx, \mathbf{p})$ by replacing $p(v) - p(x)$ with $p(y) - p(x)$, $[p(v) - p(y)]^\perp$ with $w^\perp$ and $[p(v) - p(z)]^\perp$ with $[p(y) - p(z)]^\perp$. We can view $R^\circ(G, \mathbf{p})$ as the limit of $R(G, \mathbf{p})$ when $p(v)$ tends to $p(y)$ in the direction of w (Figure 6.8(b)). In the linear matroid defined on the rows of $R^\circ(G + xy, \mathbf{p})$, the rows corresponding to vx, xy and the first row of the submatrix corresponding to vy form a circuit. Therefore, $\operatorname{rank} R^{\circ}(G,\mathbf{p}) = \operatorname{rank} R^{\circ}(G+xy,\mathbf{p}) = \operatorname{rank} R^{\circ}(G+xy-vx,\mathbf{p})$. We have

$$R^{\circ}(G+xy-vx,\mathbf{p}) = \begin{array}{c} \begin{array}{cc} v & \phantom{R(H,\mathbf{p})} \end{array} \\ \begin{pmatrix} D & * \\ 0 & R(H,\mathbf{p}) \end{pmatrix} \end{array},$$

where

$$D = \begin{pmatrix} [p(y)-p(z)]^{\perp} \\ w^{\perp} \end{pmatrix}$$

is a full column rank matrix. Therefore, $\operatorname{rank} R^{\circ}(G+xy-vx,\mathbf{p}) \geq \operatorname{rank} R(H,\mathbf{p}) + d = d|V(G)| - d$, since H is generically locally rigid. This implies $\operatorname{rank} R(G,\mathbf{p}) \geq d|V(G)| - d$ for some $\mathbf{p}$ with $p(v)$ close enough to $p(y)$. It follows from Theorem 6.4.1 that G is generically locally rigid.

<u>Case 2.4:</u> vx, vz are length edges and vy is a direction edge. The proof is similar to that of Case 2.3. ☐

1-extensions on a direction edge are more complicated as the deletion of a direction edge can reduce the rank of the rigidity matrix up to $d-1$. However, if the rank decreases by at most 1, an argument similar to the one above gives the following.

Proposition 6.4.5. *Let e be a direction edge of a generically locally rigid direction-length graph H and G be a d-dimensional 1-extension of H on e. Suppose that the deletion of e decreases the rank of the rigidity matrix of a generic realization of H by at most 1. Then G is generically locally rigid in $\mathbb{R}^d$.* ☐

Without the condition on the decrease of the rank, we have the next result for 1-extensions with two new direction edges incident to the vertices of the deleted edge.

Proposition 6.4.6. *Let H be a generically locally rigid direction-length graph in $\mathbb{R}^d$ and xy be a direction edge of H. Suppose that $G = H - xy + vx + vy + vz$ is a 1-extension of H on xy where vx, vy are direction edges. Then G is generically locally rigid in $\mathbb{R}^d$.*

Proof. Let $\mathbf{p}$ be an embedding of G such that $\mathbf{p}$ is generic on $V(H)$ and $p(v)$ is the midpoint of the segment $[p(x), p(y)]$. We can take $[p(v)-p(x)]^{\perp} = [p(v)-p(y)]^{\perp} = [p(x)-p(y)]^{\perp}$. Then in the linear matroid defined on the rows of $R(G+xy,\mathbf{p})$, the ith rows of the submatrices indexed by vx, vy and xy form a circuit, for each $i = 1, \ldots, n$. Hence, $\operatorname{rank} R(G, \mathbf{p}) = \operatorname{rank} R(G + xy, \mathbf{p}) = \operatorname{rank} R(G - vx + xy, \mathbf{p})$. As in the proof of Proposition 6.4.4, it is not difficult to show that this rank is equal to $\operatorname{rank} R(H, \mathbf{p}) + d = d|V(G)| - d$ since H is generically locally rigid. Therefore, G is generically locally rigid. $\square$

6.4.4 Union of two spanning trees

In this section we use the results in Section 6.4.2 and 6.4.3 to show the generic local rigidity in all dimensions of a special class of direction-length graphs.

Theorem 6.4.7. *Let $G = (V, D \cup L)$ be a direction-length graph such that both (V, D) and (V, L) are spanning trees. Then G is generically locally rigid in all dimensions.*

Proof. We prove the theorem by induction on the number of vertices $|V|$. The theorem is trivial when $|V| = 2$. Suppose that the theorem holds for every direction-length graph with at most $|V| - 1$ vertices. Since G is the union of two spanning trees, a simple count shows that there is a vertex v of G of degree 2 or 3.

Case 1: *G has a vertex v of degree* 2.

Then it is obvious that $G - v$ is the union of a length spanning tree and a direction spanning tree on $V - v$. Thus $G - v$ is generically locally rigid by induction hypothesis. The direction-length graph G is obtained from $G - v$ by adding a vertex v with a length edge and a direction edge connecting v to $V - v$, that is, a d-dimensional 0-extension, for every d. Therefore, by Proposition 6.4.3 G is generically locally rigid in $\mathbb{R}^d$ for every $d \geq 1$.

Case 2: *G has a vertex v of degree* 3 *and v is incident to two length edges vv_1, vv_2 and a direction edge vv_3.*

As L is the edge set of a spanning tree and $vv_1, vv_2 \in L$, the length edge $e = v_1v_2$ is not in L. Then $L - vv_1 - vv_2 + e$ is the edge set of a length spanning tree on $V - v$. We also have that $D - vv_2$ is the edge set of a direction spanning tree on $V - v$. By induction, $H = G - v + e$ is a generically locally rigid direction-length graph. Since G is a d-dimensional 1-extension of H on a length edge, G is generically locally rigid by Proposition 6.4.4.

Case 3: *G has a vertex v of degree* 3 *and v is incident to two direction edges vv_1, vv_2 and a length edge vv_3.*

As D is the edge set of a spanning tree and $vv_1, vv_2 \in D$, the direction edge $e = v_1v_2$ is not in D. Then $D - vv_1 - vv_2 + e$ is the edge set of a direction spanning tree on $V - v$. We also have $L - vv_3$ is the edge set of a spanning tree on $V - v$. By induction, $H = G - v + e$ is a generically locally rigid direction-length graph. Since G is a d-dimensional 1-extension of H on a direction edge e with new vertex v and two direction edges vv_1, vv_2 incident to the vertices of e, G is generically locally rigid by Proposition 6.4.6. $\square$

An immediate consequence of this theorem is that if a direction-length graph G contains a spanning direction tree and a spanning length tree then it is generically locally rigid in $\mathbb{R}^d$ for every d.

Chapter 7

Global rigidity of direction-length frameworks

Contents

7.1 Introduction

In Section 6.4 we have investigated the effect of 1-extensions on the local rigidity of direction-length frameworks. This chapter is devoted to the study of the effect of 1-extensions on the global rigidity of generic direction-length frameworks. A direction-length framework $(G, \mathbf{p})$ is *globally rigid* if every direction-length framework equivalent to $(G, \mathbf{p})$ is congruent to $(G, \mathbf{p})$.

In the sequel of this chapter, we will restrict our attention to algebraically generic direction-length frameworks, which we will call *generic* for short. A framework $(G, \mathbf{p})$ is (algebraically) *generic* if the set containing all coordinates of its vertices is algebraically independent over the rationals. A direction-length graph G is generically globally rigid in $\mathbb{R}^d$ if every algebraically generic realization of G in $\mathbb{R}^d$ is globally rigid.

As we have seen in Section 6.4, the local rigidity of direction-length frameworks is a generic property (Theorem 6.4.1). Meanwhile, whether global rigidity of direction-length frameworks is a generic property or not is an open problem even in dimension 2. For direction frameworks, where only directions are of interest, it is known that both local rigidity and global rigidity are generic properties [109]. Furthermore, a direction graph is generically globally rigid if and only if it is generically locally rigid. The question if global rigidity is a generic property for bar-and-joint frameworks is answered affirmatively for general dimension by Gortler, Healy and Thurston [44] with an algebraic geometry approach and, earlier, for dimension 2, by Jackson and Jordán [58] with a combinatorial constructive proof. The proof of Gortler, Healy and Thurston employed stress matrices for bar-and-joint frameworks, a notion that seems to be difficult to extend to direction-length frameworks, while the constructive proof by Jackson and Jordán is more likely extendable. In fact, Jackson and Jordán showed that the underlying graph of a globally rigid 2-dimensional generic bar-and-joint framework can be contructed from K_4 by a sequence of 1-extensions and edge-additions, two operations preserving the generic global rigidity of graphs. In [61], they explored the global rigidity preservingness of 1-extension for direction-length frameworks in $\mathbb{R}^2$. In this chapter, we extend the result of Jackson and Jordán in [61] to general dimension.

Recall from Section 6.4 that a d-dimensional *length-pure* 1*-extension* of a direction-length H deletes a length edge v_1v_2 of H, adds a new vertex v then connects v to v_1, v_2 and $d-1$ other vertices of H by length edges (Figure 6.5). A d-dimensional *direction-pure* 1*-extension* of H deletes one direction edge v_1v_2 of H, adds a new vertex v then connects v to v_1, v_2 and one other vertex of H by direction edges (Figure 6.6). A d-dimensional *reduced mixed* 1-*extension* of H the operation that deletes an edge v_1v_2 of H, adds a new vertex v then connects v to vertices in H by 3 edges e_1, e_2, e_3 such that e_1 is incident to v_1, e_2 is incident to v_2 and there is at least one direction edge and one length edge in $\{e_1, e_2, e_3\}$ (Figure 6.7). When restricted to dimension 2 these operations agree with 1-extensions for direction-length frameworks introduced by Servatius and Whiteley [94]. Jackson and Jordán proved the following result.

Theorem 7.1.1 ([61]). *Let H be a direction-length graph with $|V(H)| \geq 3$ and G a direction-length graph obtained from H by a 2-dimensional 1-extension on an edge e. Suppose that H is generically globally rigid and $H - e$ is rigid in $\mathbb{R}^2$. Then G is generically globally rigid in $\mathbb{R}^2$.*

Contribution: In using the same algebraic approach as Jackson and Jordán in [61] with a modified technique, we extend their above result to all dimensions. Our technique results in a simpler calculation in the proof for the case $d = 2$.

Theorem 7.1.2. *Let H be a direction-length graph with $|V(H)| \geq d+1$ and G a direction-length graph obtained from H by either a d-dimensional length-pure 1-extension, direction-pure 1-extension or a reduced mixed 1-extension on an edge e. Suppose that H is generically globally rigid and $H - e$ is generically locally rigid in $\mathbb{R}^d$. Then G is generically globally rigid in $\mathbb{R}^d$.*

This result provides a tool to verify the generic global rigidity of certain direction-length graphs. It can be also used for generating families of generically globally rigid direction-length graphs.

The results in this chapter are from our paper [86].

7.2 Quasi-generic direction-length frameworks

To prepare for the proof of our main theorem, in this section, we introduce and summarize some results on quasi-generic direction-length frameworks. A *quasi-generic* direction-length framework is simply a translation of a generic direction-length framework. It is convenient to work with a quasi-generic direction-length framework with one vertex coinciding with the origin. The following lemma is trivial.

Lemma 7.2.1. *Let $(G, \mathbf{p})$ be a direction-length framework with $V = V(G) = \{v_0, v_1, \ldots, v_n\}$, $p(v_0) = (0, \ldots, 0)$ and $p(v_i) = (p_{d(i-1)+1}, \ldots, p_{di})$ for $1 \leq i \leq n$. Then $(G, \mathbf{p})$ is quasi-generic if and only if $\{p_1, p_2, \ldots, p_{dn}\}$ is algebraically independent over the rationals.*

For a point $p \in \mathbb{R}^n$, we use $\mathbb{Q}(p)$ to denote the field extension of the rational field $\mathbb{Q}$ by coordinates of p. Let K be a field. We denote by $\overline{K}$ the algebraic closure of K. We will need the following lemmas.

Lemma 7.2.2 ([61])**.** *Let f_i, g_i be non-zero polynomials in n variables with rational coefficients, for $1 \leq i \leq m$. Let $T_i = \{x \in \mathbb{R}^n : g_i(x) \neq 0\}$ for $1 \leq i \leq m$ and put $T = \bigcap_{i=1}^m T_i$. Let $f : T \to \mathbb{R}^m$ defined by $f(x) = (r_1(x), r_2(x), \ldots, r_m(x))$ where $r_i(x) = f_i(x)/g_i(x)$ for $1 \leq i \leq m$.*

1. *Suppose that $\max_{x \in \mathbb{R}^n}\{\operatorname{rank} \mathrm{d}f|_x\} = m$. If p is a generic point in $\mathbb{R}^n$ then $p \in T$ and $f(p)$ is a generic point in $\mathbb{R}^m$.*

2. *Suppose that $m = n$ and $f(p)$ is a generic point in $\mathbb{R}^n$ for some $p \in T$. Then p is generic and $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f(p))}$.*

Let $(G, \mathbf{p})$ be a direction-length framework in $\mathbb{R}^d$, where $G = (V, D \cup L)$ and $D \cup L = \{e_1, e_2, \ldots, e_m\}$. We choose a reference orientation for edges in G. Let $l_{\mathbf{p}} : L \to \mathbb{R}$ defined by

$$l_{\mathbf{p}}(e) = \|p(u) - p(v)\|^2 \quad \text{for } e = uv \in L.$$

Let $d_{\mathbf{p}} : D \to \mathbb{R}^{d-1}$ be defined by

$$d_{\mathbf{p}}(e) = \left(\frac{p_1(u) - p_1(v)}{p_d(u) - p_d(v)}, \ldots, \frac{p_{d-1}(u) - p_{d-1}(v)}{p_d(u) - p_d(v)} \right) \qquad \text{if } e = uv \in D,$$

where $p_i(u)$ denotes the ith coordinate of $p(u)$ for every $1 \leq i \leq d$, assuming that this map is well-defined. Let $r_{\mathbf{p}}(e) = l_{\mathbf{p}}(e)$ if $e \in L$ and $r_{\mathbf{p}}(e) = d_{\mathbf{p}}(e)$ if $e \in D$. Putting all the components of $r_{\mathbf{p}}(e_1), r_{\mathbf{p}}(e_2), \ldots, r_{\mathbf{p}}(e_m)$ consecutively to obtain a vector $f_G(\mathbf{p})$, we call f_G the *rigidity map* of the direction-length graph G. Then it is easy to see that two realizations $(G, \mathbf{p})$ and $(G, \mathbf{q})$ of G are equivalent if and only if $f_G(\mathbf{p}) = f_G(\mathbf{q})$, assuming that the rigidity map is well-defined at both $\mathbf{p}$ and $\mathbf{q}$. It is also known that a generic direction-length framework $(G, \mathbf{p})$ is locally rigid if and only if $\operatorname{rank} \mathrm{d}f_G|_{\mathbf{p}} = d|V| - d$ (see [64, Lemma 8.1]).

Lemma 7.2.3. *Let $(G, \mathbf{p})$ be a locally rigid quasi-generic direction-length framework in $\mathbb{R}^d$ with $V = \{v_0, v_1, \ldots, v_n\}$, $p(v_0) = (0, 0, \ldots, 0)$, and $p(v_i) = (p_{d(i-1)+1}, p_{d(i-1)+2}, \ldots, p_{di})$ for $1 \leq i \leq n$. Then $(p_1, p_2, \ldots, p_{dn})$ is generic and $\overline{\mathbb{Q}(\mathbf{p})} = \overline{\mathbb{Q}(f_G(\mathbf{p}))}$.*

Proof. Assume that $(G, \mathbf{p})$ is locally rigid and is a translation of a generic direction-length framework $(G, \mathbf{q})$. Then $(G, \mathbf{q})$ is locally rigid, so $\operatorname{rank} \mathrm{d}f_G|_{\mathbf{q}} = d|V| - d$. Therefore, there exists a projection f'_G of f_G such that f'_G has $d|V| - d$ components and $\operatorname{rank} \mathrm{d}f'_G|_{\mathbf{q}} = d|V| - d$. From Lemma 7.2.2 (1) we have that $f'_G(\mathbf{q})$ is generic, thus $f'_G(\mathbf{p})$ is generic. Applying Lemma 7.2.2 (2), we have that $(p_1, p_2, \ldots, p_{dn})$ is generic and $\overline{\mathbb{Q}(\mathbf{p})} = \overline{\mathbb{Q}(f_G(\mathbf{p}))}$. □

The following corollary of Lemma 7.2.3 is crucial in the proof of our main theorem.

Corollary 7.2.4. *Suppose that $(G, \mathbf{p})$ is a locally rigid quasi-generic direction-length framework, $(G, \mathbf{q})$ is equivalent to $(G, \mathbf{p})$, and $p(v_0) = q(v_0) = (0, 0, \ldots, 0)$. Then $(G, \mathbf{q})$ is quasi-generic and $\overline{\mathbb{Q}(\mathbf{p})} = \overline{\mathbb{Q}(\mathbf{q})}$.*

7.3 Proof of Theorem 7.1.2

Let $(G, \mathbf{p})$ be a generic direction-length framework and $u, v \in V(G)$. The pair $\{u, v\}$ is said to be *globally distance linked* (*globally direction linked*, resp.) in $(G, \mathbf{p})$ if $l_{\mathbf{p}}(uv) = l_{\mathbf{q}}(uv)$ ($d_{\mathbf{p}}(uv) = d_{\mathbf{q}}(uv)$, resp.) for every framework $(G, \mathbf{q})$ equivalent to $(G, \mathbf{p})$. The pair $\{u, v\}$ is said to be *globally linked* in $(G, \mathbf{p})$ if it is both globally direction linked and globally distance linked in $(G, \mathbf{p})$. The pair $\{u, v\}$ is *generically globally distance linked* (*generically globally direction linked* or *generically globally linked*, resp.) in G if it is globally distance linked (globally direction linked or globally linked, resp.) in every generic realization of G.

Theorem 7.1.2 is just a corollary of the following theorem.

Theorem 7.3.1. *Let H be a direction-length graph with $|V(H)| \geq d + 1$ and G be a d-dimensional 1-extension of H with new vertex v. Suppose that $G - v$ is generically locally rigid.*

(a) *If G is a length-pure 1-extension of H then every pair of neighbors of v is generically globally distance linked in G.*

(b) *If G is a direction-pure 1-extension of H then every pair of neighbors of v is generically globally direction linked in G.*

(c) *If G is a reduced mixed 1-extension of H then every pair of neighbors of v is generically globally linked in G.*

Proof. Let $V = V(G) = \{v_0, v_1, \ldots, v_n\}$ and suppose that $v = v_n$. Let $(G, \mathbf{p})$ be a quasi-generic realization of G and $(G, \mathbf{q})$ an equivalent realization of $(G, \mathbf{p})$. Assume that

$$\begin{aligned} p(v_0) = q(v_0) &= (0, 0, \ldots, 0), \\ p(v_i) &= (p_{d(i-1)+1}, p_{d(i-1)+2}, \ldots, p_{di}) \qquad \text{for } 1 \le i \le n, \\ q(v_i) &= (q_{d(i-1)+1}, q_{d(i-1)+2}, \ldots, q_{di}) \qquad \text{for } 1 \le i \le n. \end{aligned}$$

By Lemma 7.2.1, $\{p_1, p_2, \ldots, p_n\}$ is algebraically independent over $\mathbb{Q}$.

Let $\mathbf{p}' = \mathbf{p}|_{V-v}$ and $\mathbf{q}' = \mathbf{q}|_{V-v}$, the restriction of $\mathbf{p}$ and $\mathbf{q}$ on the vertex set of H. Then $(G - v, \mathbf{q}')$ is equivalent to $(G - v, \mathbf{p}')$ and $(G - v, \mathbf{p}')$ is quasi-generic. Since $G - v$ is generically locally rigid, Corollary 7.2.4 implies that $(G - v, \mathbf{q}')$ is quasi-generic and $\overline{\mathbb{Q}(\mathbf{p}')} = \overline{\mathbb{Q}(\mathbf{q}')}$. In particular, the set $\{q_1, q_2, \ldots, q_{d(n-1)}\}$ is algebraically independent over the rationals. Let $K = \overline{\mathbb{Q}(\mathbf{p}')} = \overline{\mathbb{Q}(\mathbf{q}')}$.

Proof of (a)

Without loss of generality, we may assume that the set of neighbors of v is $\{v_0, v_1, \ldots, v_d\}$. Since $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent, we have the following equations

$$q_{d(n-1)+1}^2 + \cdots + q_{dn}^2 = p_{d(n-1)+1}^2 + \cdots + p_{dn}^2, \tag{0}$$

$$(q_{d(n-1)+1} - q_1)^2 + \cdots + (q_{dn} - q_d)^2 = (p_{d(n-1)+1} - p_1)^2 + \cdots + (p_{dn} - p_d)^2, \tag{1}$$

$$\vdots$$

$$(q_{d(n-1)+1} - q_{d(d-1)+1})^2 + \cdots + (q_{dn} - q_{d^2})^2 = (p_{d(n-1)+1} - p_{d(d-1)+1})^2 + \cdots + (p_{dn} - p_{d^2})^2. \tag{d}$$

Subtracting equations (1), ..., (d) from equation (0), we have the following system of linear equations:

$$A_q q(v_n)^T = A_p p(v_n)^T + \mathbf{a},$$

where A_p, A_q are $d \times d$ matrices with all components belong to K and $\mathbf{a}$ is a vector with d components in K. In fact,

$$A_q = \begin{pmatrix} q_1 & q_2 & \cdots & q_d \\ q_{d+1} & q_{d+2} & \cdots & q_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ q_{d(d-1)+1} & q_{d(d-1)+2} & \cdots & q_{d^2} \end{pmatrix}$$

is a non-singular matrix since $\{q_1, q_2, \ldots, q_{d(n-1)}\}$ is algebraically independent over the rationals, and

$$\mathbf{a} = \frac{1}{2} \times \begin{pmatrix} \|q(v_1)\|^2 - \|p(v_1)\|^2 \\ \|q(v_2)\|^2 - \|p(v_2)\|^2 \\ \vdots \\ \|q(v_d)\|^2 - \|p(v_d)\|^2 \end{pmatrix}.$$

Therefore, $q(v_n)^T = A_q^{-1} A_p p(v_n)^T + A_q^{-1}\mathbf{a}$. In particular, $q_{d(n-1)+1}, \ldots, q_{dn} \in K(p_{d(n-1)+1}, \ldots, p_{dn})$.

Since (0) holds, the constant term of each q_j, $d(n-1)+1 \leq j \leq dn$, regarded as polynomial of $p_{d(n-1)+1}, \ldots, p_{dn}$ in K, is 0. Thus, $A_q^{-1}\mathbf{a} = 0$, which implies that $\mathbf{a} = 0$. Therefore $\{v_0, v_1\}, \{v_0, v_2\}, \ldots, \{v_0, v_d\}$ are globally distance linked in G. Symmetry implies that all pairs of neighbors of v are globally distance linked in G.

Proof of (b)

Without loss of generality we can assume that the set of neighbors of v is $\{v_0, v_1, v_2\}$. Since $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent, we have the following systems of equations

$$\begin{aligned} \frac{q_{dn-1}}{q_{dn}} &= \frac{p_{dn-1}}{p_{dn}}, \\ \frac{q_{dn-1} - q_{d-1}}{q_{dn} - q_d} &= \frac{p_{dn-1} - p_{d-1}}{p_{dn} - p_d}, \\ \frac{q_{dn-1} - q_{2d-1}}{q_{dn} - q_{2d}} &= \frac{p_{dn-1} - p_{2d-1}}{p_{dn} - p_{2d}}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} q_{dn-1}p_{dn} &= q_{dn}p_{dn-1}, && \text{(i)} \\ (q_{dn-1} - q_{d-1})(p_{dn} - p_d) &= (q_{dn} - q_d)(p_{dn-1} - p_{d-1}), && \text{(ii)} \\ (q_{dn-1} - q_{2d-1})(p_{dn} - p_{2d}) &= (q_{dn} - q_{2d})(p_{dn-1} - p_{2d-1}). && \text{(iii)} \end{aligned}$$

Subtracting equations (ii) and (iii) from (i), we obtain the following linear system of equations

$$A(q_{dn-1}, q_{dn})^T = B(p_{dn-1}, p_{dn})^T + \mathbf{c},$$

where

$$A = \begin{pmatrix} p_d & -p_{d-1} \\ p_{2d} & -p_{2d-1} \end{pmatrix}$$

is a non-singular matrix, and

$$\mathbf{c} = \begin{pmatrix} q_{d-1}p_d - q_d p_{d-1} \\ q_{2d-1}p_{2d} - q_{2d}p_{2d-1} \end{pmatrix}.$$

Therefore, $(q_{dn-1}, q_{dn})^T = A^{-1}B(p_{dn-1}, p_{dn})^T + A^{-1}\mathbf{c}$. Substituting this into (i), we obtain a polynomial of p_{dn}, p_{dn-1} over K which must be identically zero. It is easy to see that the coefficient of p_{dn} in this polynomial is the first component of $A^{-1}\mathbf{c}$ and the coefficient of p_{dn-1} is equal to the negative of the second component of $A^{-1}\mathbf{c}$. Thus $A^{-1}\mathbf{c}$ must be zero, which implies that $\mathbf{c} = \mathbf{0}$. Therefore,

$$\frac{q_{d-1}}{q_d} = \frac{p_{d-1}}{p_d},$$

and

$$\frac{q_{2d-1}}{q_{2d}} = \frac{p_{2d-1}}{p_{2d}}$$

hold. By symmetry we obtain similar equalities for other components and other pairs of neighbors of v. Therefore, every pair of neighbors of v is globally linked in G.

Proof of (c)

(c-1) We consider the case that the mixed 1-extension has one length edge $v_n v_0$ and two direction edges $v_n v_1, v_n v_2$.

Using the fact that $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent, we have

$$q_{d(n-1)+1}^2 + q_{d(n-1)+2}^2 + \cdots + q_{dn}^2 = p_{d(n-1)+1}^2 + p_{d(n-1)+2}^2 + \cdots + p_{dn}^2, \tag{0'}$$

$$(q_{dn} - q_d)(p_{d(n-1)+1} - p_1) = (q_{d(n-1)+1} - q_1)(p_{dn} - p_d), \tag{1'}$$

$$\vdots$$

$$(q_{dn} - q_d)(p_{dn-1} - p_{d-1}) = (q_{dn-1} - q_{d-1})(p_{dn} - p_d), \tag{(d-1)'}$$

$$(q_{dn} - q_{2d})(p_{dn-1} - p_{2d-1}) = (q_{dn-1} - q_{2d-1})(p_{dn} - p_{2d}). \tag{d'}$$

We rewrite the system of equations from (1') to (d') as

$$Xq(v_n)^T = Yp(v_n)^T + \mathbf{z},$$

where

$$X = \begin{pmatrix} p_{dn} - p_d & & \mathbf{0} & -(p_{d(n-1)+1} - p_1) \\ & \ddots & & \vdots \\ \mathbf{0} & & p_{dn} - p_d & -(p_{dn-1} - p_{d-1}) \\ & & p_{dn} - p_{2d} & -(p_{dn-1} - p_{2d-1}) \end{pmatrix},$$

and

$$\mathbf{z} = \begin{pmatrix} p_1q_d - p_dq_1 \\ \vdots \\ p_{d-1}q_d - p_dq_{d-1} \\ p_{2d-1}q_{2d} - p_{2d}q_{2d-1} \end{pmatrix}.$$

Let X_i be the matrix obtained from X by replacing the i-th column of X with $Yp(v_n)^T + \mathbf{z}$. Let $D = \det X$, $D_i = \det X_i$ for $1 \leq i \leq d$. Then $q_{d(n-1)+i} = D_i/D$ for $1 \leq i \leq d$ by Cramer's rule. Substituting into (0'), we have

$$D_1^2 + \cdots + D_d^2 = D^2(p_{d(n-1)+1}^2 + \cdots + p_{dn}^2) \tag{$*$}$$

with $D, D_i \in K(p_{d(n-1)+1}, \ldots, p_{dn-1}, p_{dn})$ for $1 \leq i \leq d$.

The constant term in the right side of $(*)$ is zero, thus the constant term α_i of each D_i must be zero. Let

$$T = \begin{pmatrix} -p_d & & \mathbf{0} & p_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & -p_d & p_{d-1} \\ & & -p_{2d} & p_{2d-1} \end{pmatrix}$$

be the matrix obtained from X by setting all $p_{d(n-1)+1}, \ldots, p_{dn-1}, p_{dn}$ to 0 (we can consider T as the matrix consisting of the constant terms of X in K). Let T_i be the matrix obtained from T by replacing the i-th column of T with $\mathbf{z}$ (equivalently, T_i is the matrix consisting of the constant terms of X_i). Then $\det T_i = 0$ for $1 \leq i \leq d$. Note that $\det T = (-p_d)^{d-2}(p_{d-1}p_{2d} - p_dp_{2d-1}) \neq 0$, Cramer's rule implies that $x = (0, \ldots, 0)$ is the only solution of the linear system $Tx = \mathbf{z}$. Hence, $\mathbf{z} = \mathbf{0}$ holds. Therefore,

$$\frac{q_d}{p_d} = \frac{q_{d-1}}{p_{d-1}} = \cdots = \frac{q_1}{p_1} = \alpha. \tag{c1}$$

Similarly, we obtain

$$\frac{q_{2d}}{p_{2d}} = \frac{q_{2d-1}}{p_{2d-1}} = \cdots = \frac{q_{d+1}}{p_{d+1}} = \beta. \tag{c2}$$

Since $\mathbf{z} = 0$, we have

$$Yp(v_n)^T + \mathbf{z} = \begin{pmatrix} q_1p_{dn} - q_dp_{d(n-1)+1} \\ \vdots \\ q_{d-1}p_{dn} - q_dp_{dn-1} \\ q_{2d-1}p_{dn} - q_{2d}p_{dn-1} \end{pmatrix}.$$

For a polynomial P in $K[X_1,\dots,X_d]$, let $\mathrm{Coeff}_P(X_1^{t_1}\dots X_d^{t_d})$ denote the coefficient of the monomial $X_1^{t_1}\dots X_d^{t_d}$ in P. Then the coefficient of p_{dn}^{2d} in the left of $(*)$ is equal to

$$\begin{aligned}\mathrm{Coeff}_{D_d^2}(p_{dn}^{2d}) &= [\mathrm{Coeff}_{D_d}(p_{dn}^{d})]^2\\ &= (q_{d-1}-q_{2d-1})^2\end{aligned}$$

and the coefficient of p_{dn}^{2d} in the right of $(*)$ is equal to

$$\begin{aligned}\mathrm{Coeff}_{D^2}(p_{dn}^{2d-2}) &= [\mathrm{Coeff}_{D}(p_{dn}^{d-1})]^2\\ &= (p_{d-1}-p_{2d-1})^2.\end{aligned}$$

Comparing these two coefficients, we obtain that $(q_{d-1}-q_{2d-1})^2=(p_{d-1}-p_{2d-1})^2$ holds. Since the role of every component is equivalent, we have $(q_d-q_{2d})^2=(p_d-p_{2d})^2$. Thus

$$q_{d-1}-q_{2d-1}=\pm(p_{d-1}-p_{2d-1}), \tag{c3}$$

and

$$q_d-q_{2d}=\pm(p_d-p_{2d}). \tag{c4}$$

We now evaluate the coefficient of $p_{dn}^{2d-1}p_{dn-1}$ in the left of $(*)$ which is equal to

$$\begin{aligned}\mathrm{Coeff}_{D_d^2}(p_{dn}^{2d-1}p_{dn-1}) &= 2\times\mathrm{Coeff}_{D_d}(p_{dn}^{d})\times\mathrm{Coeff}_{D_d}(p_{dn}^{d-1}p_{dn-1})\\ &= -2(q_{d-1}-q_{2d-1})(q_d-q_{2d}),\end{aligned}$$

and the coefficient of $p_{dn}^{2d-1}p_{dn-1}$ in the right of $(*)$ which is equal to

$$\begin{aligned}\mathrm{Coeff}_{D^2}(p_{dn}^{2d-3}p_{dn-1}) &= 2\times\mathrm{Coeff}_{D}(p_{dn}^{d-1})\times\mathrm{Coeff}_{D}(p_{dn}^{d-2}p_{dn-1})\\ &= -2(p_{d-1}-p_{2d-1})(p_d-p_{2d}).\end{aligned}$$

Comparing these two coefficients, we have $(q_{d-1}-q_{2d-1})(q_d-q_{2d})=(p_{d-1}-p_{2d-1})(p_d-p_{2d})$. Together with (c3) and (c4) this equation implies that, if $q_{d-1}-q_{2d-1}=p_{d-1}-p_{2d-1}$ then $q_d-q_{2d}=p_d-p_{2d}$, and if $q_{d-1}-q_{2d-1}=-(p_{d-1}-p_{2d-1})$ then $q_d-q_{2d}=-(p_d-p_{2d})$. Suppose that $q_{d-1}-q_{2d-1}=p_{d-1}-p_{2d-1}$ and $q_d-q_{2d}=p_d-p_{2d}$. Using (c1) and (c2) we have,

$$\alpha p_{d-1}-\beta p_{2d-1}=p_{d-1}-p_{2d-1},$$

$$\alpha p_d-\beta p_{2d}=p_d-p_{2d}.$$

Since $\{p_{d-1},p_d,p_{2d-1},p_{2d}\}$ is algebraically independent over the rationals, we obtain $\alpha=\beta=1$, which implies that $p(v_1)=q(v_1)$ and $p(v_2)=p(v_1)$, i.e., the pairs $\{v_0,v_1\},\{v_0,v_2\},\{v_1,v_2\}$ are globally linked in G.

Similar argument can be applied for the case $q_{d-1} - q_{2d-1} = -(p_{d-1} - p_{2d-1})$ and $q_d - q_{2d} = -(p_d - p_{2d})$ to deduce that $p(v_1) = -q(v_1)$ and $p(v_2 = -q(v_2)$.

The special case that v has just two neighbors in $G - v$, i.e., v is connected to v_0 by one length edge and one direction edge, and v is connected to v_1 by one length edge can be treated similarly as above by putting $q(v_2) = p(v_2) = \mathbf{0}$.

(c-2) We now consider the case that the mixed 1-extension has one direction edge $v_n v_0$ and two length edges $v_n v_1, v_n v_2$.

Since $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent we have the following equations

$$\begin{gathered}
\frac{q_{d(n-1)+1}}{q_{dn}} = \frac{p_{d(n-1)+1}}{p_{dn}}, \\
\vdots \\
\frac{q_{dn-1}}{q_{dn}} = \frac{p_{dn-1}}{p_{dn}}, \\
(q_{d(n-1)+1} - q_1)^2 + \cdots + (q_{dn} - q_d)^2 = (p_{d(n-1)+1} - p_1)^2 + \cdots + (p_{dn} - p_d)^2, \\
(q_{d(n-1)+1} - q_{d+1})^2 + \cdots + (q_{dn} - q_{2d})^2 = (p_{d(n-1)+1} - p_{d+1})^2 + \cdots + (p_{dn} - p_{2d})^2.
\end{gathered}$$

From the two last equations we derive a linear equation of $q(v_n)$. Together with the first $d - 1$ equations we obtain,

$$Mq(v_n)^T = \mathbf{r},$$

where

$$M = \begin{pmatrix} p_{dn} & & \mathbf{0} & -p_{d(n-1)+1} \\ & \ddots & & \vdots \\ \mathbf{0} & & p_{dn} & -p_{dn-1} \\ q_1 - q_{d+1} & \dots & q_{d-1} - q_{2d-1} & q_d - q_{2d} \end{pmatrix},$$

$$\mathbf{r} = \begin{pmatrix} \mathbf{0} \\ s + r \end{pmatrix},$$

with

$$s = (p_1 - p_{d+1})p_{d(n-1)+1} + \cdots + (p_d - p_{2d})p_{dn},$$

and

$$r = \frac{1}{2}(\|q(v_1)\|^2 - \|p(v_1)\|^2 - \|q(v_2)\|^2 + \|p(v_2)\|^2).$$

Let M_i be the matrix obtained from M by replacing the i-th column of M with $\mathbf{r}$. Let $F = \det M$ and $F_i = \det M_i$ for $1 \leq i \leq d$. Then $F_i = (s + r)p_{dn}^{d-2} p_{d(n-1)+i}$ for $1 \leq i \leq d$. In particular, $F_d = (s + r)p_{dn}^{d-1}$. We can observe that the highest degree of p_{dn} in F, considered as a polynomial over K, is $d - 1$.

Solving the linear system for $q(v_n)$ we obtain $q_{d(n-1)+i} = F_i/F$ for $i = 1, \ldots, d$. We now rewrite the equation for $v_n v_1$ as

$$(F_1 - q_1 F)^2 + \cdots + (F_d - q_d F)^2 = F^2[(p_{d(n-1)+1} - p_1)^2 + \cdots + (p_{dn} - p_d)^2]. \quad (**)$$

We first evaluate the coefficient of p_{dn}^{2d} in the left of $(**)$ which is equal to

$$\begin{aligned} \mathrm{Coeff}_{F_d^2}(p_{dn}^{2d}) &= [\mathrm{Coeff}_{F_d}(p_{dn}^{d})]^2 \\ &= (p_d - p_{2d})^2, \end{aligned}$$

and the coefficient of p_{dn}^{2d} in the right of $(**)$ which is equal to

$$\begin{aligned} \mathrm{Coeff}_{F^2}(p_{dn}^{2d-2}) &= [\mathrm{Coeff}_{F}(p_{dn}^{d-1})]^2 \\ &= (q_d - q_{2d})^2. \end{aligned}$$

Thus, by comparing these two coefficients,

$$(p_d - p_{2d})^2 = (q_d - q_{2d})^2 \quad \text{(c5)}$$

holds.

We then evaluate the coefficient of p_{dn}^{2d-2} in the left of $(**)$, which is

$$\begin{aligned} \mathrm{Coeff}_{F_d^2}(p_{dn}^{2d-2}) + (q_1^2 + \cdots + q_d^2) \times \mathrm{Coeff}_{F^2}(p_{dn}^{2d-2}) - 2q_d \times \mathrm{Coeff}_{F_d}(p_{dn}^{d-1}) \times \mathrm{Coeff}_{F}(p_{dn}^{d-1}) \\ = r^2 + \|q(v_1)\|^2 (q_d - q_{2d})^2 - 2q_d r(q_d - q_{2d}), \end{aligned}$$

and the coefficient of p_{dn}^{2d-2} in the right of $(**)$

$$(p_1^2 + \cdots + p_d^2)\mathrm{Coeff}_{F^2}(p_{dn}^{2d-2}) = \|p(v_1)\|^2 (q_d - q_{2d})^2.$$

It follows that

$$r^2 - 2q_d r(q_d - q_{2d}) = (\|p(v_1)\|^2 - \|q(v_1)\|^2)(q_d - q_{2d})^2. \quad \text{(c6)}$$

We also evaluate the coefficient of p_{dn}^{2d-1} in the left of $(**)$ which is

$$\begin{aligned} -2q_d \times \mathrm{Coeff}_{F_d}(p_{dn}^{d}) \times \mathrm{Coeff}_{F}(p_{dn}^{d-1}) + \mathrm{Coeff}_{F^2}(p_{dn}^{2d-1}) \\ = -2q_d(p_d - p_{2d})(q_d - q_{2d}) + 2r(p_d - p_{2d}), \end{aligned}$$

and the coefficient of p_{dn}^{2d-1} in the right of $(**)$ which is

$$\begin{aligned} -2p_d \times \mathrm{Coeff}_{F^2}(p_{dn}^{2d-1}) &= -2p_d(q_d - q_{2d})^2 \\ &= -2p_d(p_d - p_{2d})^2 \quad \text{(by (c5))}. \end{aligned}$$

Therefore, $-2q_d(p_d - p_{2d})(q_d - q_{2d}) + 2r(p_d - p_{2d}) = -2p_d(p_d - p_{2d})^2$, which is equivalent to $r - q_d(q_d - q_{2d}) = -p_d(p_d - p_{2d})$. Squaring both sides of this equation, we have $r^2 - 2rq_d(q_d - q_{2d}) + q_d^2(q_d - q_{2d})^2 = p_d^2(p_d - p_{2d})^2$. Using (c6) we have $(\|p(v_1)\|^2 - \|q(v_1)\|^2)(q_d - q_{2d})^2 + q_d^2(q_d - q_{2d})^2 = p_d^2(p_d - p_{2d})^2$. Since $(q_d - q_{2d})^2 = (p_d - p_{2d})^2$ by (c5), this implies that $\|p(v_1)\|^2 - \|q(v_1)\|^2 = p_d^2 - q_d^2$. This equation must hold for every coordinate due to the equivalence of their roles. It follows that $d(\|p(v_1)\|^2 - \|q(v_1)\|^2) = (p_1^2 - q_1^2) + \cdots + (p_d^2 - q_d^2) = \|p(v_1)\|^2 - \|q(v_1)\|^2$. Therefore $\|p(v_1)\|^2 = \|q(v_1)\|^2$ holds. Similarly, $\|p(v_2)\|^2 = \|q(v_2)\|^2$ holds. Thus $r = 0$ holds, then $q_d(q_d - q_{2d}) = p_d(p_d - p_{2d})$ holds. Together with (c5), this implies that

$$\frac{p_d}{q_d} = \frac{p_d - p_{2d}}{q_d - q_{2d}} = \frac{p_{2d}}{q_{2d}} = \gamma = \pm 1 \tag{c7}$$

holds. Similarly,

$$\frac{p_{d-1}}{q_{d-1}} = \frac{p_{d-1} - p_{2d-1}}{q_{d-1} - q_{2d-1}} = \frac{p_{2d-1}}{q_{2d-1}} = \delta = \pm 1 \tag{c8}$$

holds.

Last, we evaluate the coefficient of $p_{dn}^{2d-1}p_{dn-1}$ in the left of $(**)$

$$\begin{aligned}\mathrm{Coeff}_{F_d^2}(p_{dn}^{2d-1}p_{dn-1}) &= 2 \times \mathrm{Coeff}_{F_d}(p_{dn}^{d}) \times \mathrm{Coeff}_{F_d}(p_{dn}^{d-1}p_{dn-1}) \\ &= 2(p_d - p_{2d})(p_{d-1} - p_{2d-1}),\end{aligned}$$

and the coefficient of $p_{dn}^{2d-1}p_{dn-1}$ in the right of $(**)$ which is

$$\begin{aligned}\mathrm{Coeff}_{F^2}(p_{dn}^{2d-3}p_{dn-1}) &= 2 \times \mathrm{Coeff}_{F}(p_{dn}^{d-1}) \times \mathrm{Coeff}_{F}(p_{dn}^{d-2}p_{dn-1}) \\ &= 2(q_d - q_{2d})(q_{d-1} - q_{2d-1}).\end{aligned}$$

Thus, by comparing these two coefficients, $(p_d - p_{2d})(p_{d-1} - p_{2d-1}) = (q_d - q_{2d})(q_{d-1} - q_{2d-1})$ holds. Together with (c7) and (c8) we have $\gamma = \delta$. By symmetry we conclude that $\{v_0, v_1\}, \{v_0, v_2\}$ and $\{v_1, v_2\}$ are globally linked in G.

The special case that v has just 2 neighbors in $G - v$, i.e., v is connected to v_0 by one direction edge and one length edge, and v is connected to v_1 by one direction edge, can be treated similarly as above by putting $q(v_2) = p(v_2) = \mathbf{0}$. $\square$

Proof of Theorem 7.1.2. Let $(G, \mathbf{p})$ be a generic realization of G in $\mathbb{R}^d$ and $(G, \mathbf{q})$ a realization in $\mathbb{R}^d$ that is equivalent to $(G, \mathbf{p})$. We show that $(G, \mathbf{q})$ is congruent to $(G, \mathbf{p})$. Let $e = v_1v_2$ and v be the new vertex. Since $G - v = H - e$ is generically locally rigid, applying Theorem 7.3.1, we have that, in $(G, \mathbf{p})$, the pair $\{v_1, v_2\}$ is globally direction linked if e is a direction edge and $\{v_1, v_2\}$ is globally distance linked if e is a length edge. Therefore, $(H, \mathbf{q})$ is equivalent to $(H, \mathbf{p})$ and hence they are congruent as H is generically globally rigid. So there exists a realization $(G, \mathbf{p}')$ congruent to $(G, \mathbf{p})$ and a realization $(G, \mathbf{q}')$ congruent to $(G, \mathbf{q})$ such that $(H, \mathbf{q}')$ coincides to $(H, \mathbf{p}')$. Then it is not difficult to see that the distance and/or direction constraints between v and its neighbors uniquely determine the position of v, which means that $\mathbf{p}'(v) = \mathbf{q}'(v)$. Therefore, $(G, \mathbf{p}')$ is congruent to $(G, \mathbf{q}')$, which implies that $(G, \mathbf{p})$ is congruent to $(G, \mathbf{q})$. $\square$

7.4 Further remarks

The assumption about the rigidity of $H - e$ is essential in our proof. This assumption is however not necessary for the analogous result to Theorem 7.1.2 for bar-and-joint frameworks [62] due to the fact that a globally rigid generic bar-and-joint framework is redundantly locally rigid [53]. The behavior of a globally rigid direction-length framework after deleting an edge is much more complicated. Some partial results on this behavior have been obtained by Jackson and Keevash.

Theorem 7.4.1 ([65]). *Suppose that G is a generically globally rigid direction-length graph in $\mathbb{R}^d$ with at least two length edges. Then $G - e$ is generically locally rigid in $\mathbb{R}^d$ for any length edge e of G.*

Combining with Theorem 7.1.2, we obtain the following consequence.

Corollary 7.4.2. *Suppose that H is a globally rigid direction-length graph in $\mathbb{R}^d$ with at least two length edges and G is obtained from H by a d-dimensional 1-extension on a length edge. Then G is generically globally rigid in $\mathbb{R}^d$.*

The suppression of a direction edge removes $d - 1$ constraints. Therefore, the behavior of a direction-length framework after deleting a direction edge must be much more complicated in dimension $d \geq 3$. Even in dimension 2, only partial results are obtained. A direction-length framework $(G, \mathbf{p})$ is said to be *bounded* if there exists a $K > 0$ such that $\|q(u) - q(v)\| < K$ for every framework $(G, \mathbf{q})$ equivalent to $(G, \mathbf{p})$. Jackson and Keevash showed the following.

Theorem 7.4.3 ([65]). *Suppose that G is a 2-dimensional generically globally rigid direction-length graph, e is a direction edge of G, and $G - e$ has a generically locally rigid subgraph with more than one vertex. Then $G - e$ is either rigid or unbounded in $\mathbb{R}^2$.*

They conjectured that the assumption about the existence of a rigid subgraph with more than one vertex of $G - e$ is not necessary.

Conjecture 7.4.4 ([64]). *Suppose that G is a 2-dimensional generic globally rigid direction-length framework with at least two length edges and e is a direction edge of G. Then $G - e$ is either rigid or unbounded in $\mathbb{R}^2$.*

Chapter 8

Universal rigidity

Contents

8.1 Introduction

First let us recall that a framework $(G, \mathbf{p})$ in $\mathbb{R}^r$ is said to be in *general position* if for every $U \subseteq V(G)$ with $|U| \geq r+1$, the set $\{p(v) : v \in U\}$ is affinely independent. In dimension one, this condition simply means that no two vertices are mapped to the same point on the line. The *affine dimension* of a framework $(G, \mathbf{p})$ is the dimension of the affine space spanned by $\{p(v) : v \in V(G)\}$. A framework $(G, \mathbf{p})$ in $\mathbb{R}^r$ is said to be (algebraically) *generic* if the set of all coordinates of $\mathbf{p}$ is algebraically independent over the rationals.

Although the first result on universal rigidity of frameworks appeared in the work of Connelly [21] more than 30 years ago, the study of universal rigidity has attracted much interest only since the last decade, when algorithms using semidefinite programming are employed to solve network localization problem [15, 95]. Even when the solution is unique in 2 or 3-dimensional space, there is no guarantee that these algorithms return this solution. On the contrary, they tend to give a solution in a higher dimension. Hence universal rigidity is a desirable property to ensure the efficiency of these algorithms. However, it is likely that universal rigidity is more difficult to characterize than local rigidity and global rigidity. Still little is known about universal rigidity, even for generic frameworks in dimension 1, where the characterizations of local and global rigidity are almost trivial. Moreover, the fact that universal rigidity is not a generic property makes it difficult to handle.

In this chapter, we study universal rigidity in two directions. The first one is to explore universal rigidity of bar-joint frameworks on the line. In contrast to local rigidity and global rigidity, the universal rigidity of frameworks seems quite complicated even in $\mathbb{R}^1$. Therefore, we start by investigating a special class of frameworks: complete bipartite frameworks. We obtain a complete characterization of the universal rigidity of these frameworks in general position on the line and deduce that the only bipartite graph that is generically universally rigid in $\mathbb{R}^d$ is $K_{1,1}$ for every $d \geq 1$. This result suggests the study of other classes of frameworks which share some properties with bipartite frameworks as well as inspires several open questions and conjectures.

The second direction is to relax the condition on the genericity of frameworks. As the study of universal rigidity arose from the quest for an efficient algorithm for network localization problem, the requirement on genericity is far from satisfactory. An earlier work of Alfakih and Ye [8] gave sufficient condition for the universal rigidity of bar-joint frameworks in general position, a weaker requirement than genericity. We strengthen the results of Alfakih and Ye for bar-joint frameworks, allowing configurations in non general position. In fact, our result is obtained for tensegrity frameworks, a generalized model of bar-joint frameworks, where beside bars we have also cables, which prevent the distance between vertices from increasing, and struts, which prevent the distance between vertices from decreasing.

8.2 Universal rigidity in $\mathbb{R}^1$

The results in this section are from a joint work with Tibor Jordán [70].

8.2.1 Universal rigidity of bipartite frameworks

Our main result is the following.

Theorem 8.2.1. *Let G be a complete bipartite graph on at least three vertices with bipartition $X = \{x_1, \dots, x_m\}$, $Y = \{y_1, \dots, y_n\}$ and $\mathbf{p}$ a realization of G in general position on the line.*

1. *If $p(x_1) < \cdots < p(x_m) < p(y_1) < \cdots < p(y_n)$, then $(G, \mathbf{p})$ is not universally rigid.*
2. *If $p(x_1) < \cdots < p(x_k) < p(y_1) < \cdots < p(y_n) < p(x_{k+1}) < \cdots < p(x_m)$ (or symmetrically $p(y_1) < \cdots < p(y_k) < p(x_1) < \cdots < p(x_m) < p(y_{k+1}) < \cdots < p(y_n)$), then $(G, \mathbf{p})$ is not universally rigid.*
3. *If none of the conditions above holds, then $(G, \mathbf{p})$ is universally rigid.*

In order to prove this theorem, we will need the following result.

Theorem 8.2.2 (Alfakih [4])**.** *Let $(G, \mathbf{p})$ be a framework in $\mathbb{R}^d$. Then $(G, \mathbf{p})$ has a non-zero positive semidefinite (PSD) stress matrix Ω if and only if $(G, \mathbf{p})$ has no equivalent realization of affine dimension $|V| - 1$.*

We will also need some more definitions. Let $(G, \mathbf{p})$ be a framework on the line with $G = (V, E)$. A pair of vertices $u, v \in V$ is called *universally linked* in $(G, \mathbf{p})$ if $\|q(u) - q(v)\| = \|p(u) - p(v)\|$ holds for all frameworks $(G, \mathbf{q})$ which are equivalent to $(G, \mathbf{p})$ (in all dimensions). Let C be a cycle of G passing through $v_1, \dots, v_k$ with $E(C) = \{v_1v_2, \dots, v_{k-1}v_k, v_kv_1\}$. If $p(v_1) < p(v_2) < \cdots < p(v_k)$ then C is called a *stretched cycle* in $(G, \mathbf{p})$ (Figure 8.1). If C is a stretched cycle in $(G, \mathbf{p})$

then it is not difficult to see that every pair of vertices of C is universally linked in $(G, \mathbf{p})$. In fact, the condition that the distance v_1v_k is the sum of the distance $v_1v_2, \ldots, v_{k-1}v_k$ implies that $v_1, \ldots, v_k$ must lie on a line and respect the order and distance between any pairs.

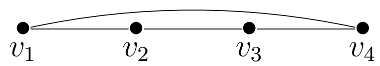


Figure 8.1: A stretched cycle on 4 vertices. The bar v_1v_4 is bent a little so that the other bars can be seen.

Now we are ready to prove Theorem 8.2.1.

Proof of Theorem 8.2.1.
1. Suppose that $p(x_1) < \cdots < p(x_m) < p(y_1) < \cdots < p(y_n)$ holds and consider a PSD stress matrix Ω of $(G, \mathbf{p})$. We will prove that Ω is the zero matrix.

Let $r_{ij} = p(y_j) - p(x_i) > 0$ denote the distance between x_i and y_j in $(G, \mathbf{p})$, and w_{ij} the stress on the edge x_iy_j, for $1 \leq i \leq m$ and $1 \leq j \leq n$. The equilibrium condition at vertices in X gives

$$\sum_j r_{ij} w_{ij} = 0, \quad \text{for every } i = 1, \ldots, m. \tag{8.2.1}$$

Let $s_j = p(y_j) - p(y_1)$ be the distance between y_1 and y_j. Then we have $r_{ij} - r_{i1} = s_j$, for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$ (see Figure 8.2.)

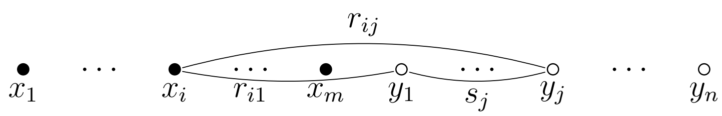


Figure 8.2: $r_{ij} - r_{i1} = s_j$

The entries on the diagonal of Ω are $\sum_{j=1}^n w_{ij}$, for $i = 1, \ldots, m$, and $\sum_{i=1}^m w_{ij}$, for $j = 1, \ldots, n$. Since Ω is PSD, these entries are all non-negative. Therefore,

$$\sum_{j=1}^n r_{i1} w_{ij} \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Using (8.2.1), we have

$$0 \leq \sum_j r_{i1} w_{ij} = \sum_j r_{i1} w_{ij} - \sum_j r_{ij} w_{ij} = \sum_j (r_{i1} - r_{ij}) w_{ij} = -\sum_{j>1} s_j w_{ij}.$$

Therefore, $\sum_{j>1} s_j w_{ij} \leq 0$, for $j = 1, \ldots, n$. Then, since $s_j > 0$ for $j = 2, \ldots, n$,

$$0 \leq \sum_{j>1} s_j \sum_i w_{ij} = \sum_i \sum_{j>1} s_j w_{ij} \leq 0$$

which implies that equality holds everywhere. Thus, all entries on the diagonal of Ω are 0's with possibly an exception of the entry corresponding to (x_1, x_1). However, by using the symmetry of the graph, we can deduce that this entry must also be 0. Therefore, the sum of all eigenvalues of Ω is 0. Hence Ω is the zero matrix. Theorem 8.2.2 now implies that $(G, \mathbf{p})$ is not universally rigid, in fact, it has an equivalent realization of affine dimension $m + n - 1$.

2. Suppose that $p(x_1) < \cdots < p(x_k) < p(y_1) < \cdots < p(y_n) < p(x_{k+1}) < \cdots < p(x_m)$ holds and consider a PSD stress matrix Ω of $(G, \mathbf{p})$. We will prove that Ω is the zero matrix.

Let r_{ij} be the distance between x_i and y_j in this realization and w_{ij} the stress on the edge $x_i y_j$. Let

$$q_j = \begin{cases} r_{ij} - r_{i1}, & \text{for } i \leq k \\ r_{i1} - r_{ij}, & \text{for } i \geq k+1 \end{cases}$$

and

$$t_i = \begin{cases} r_{k+1,j} + r_{ij}, & \text{for } i \leq k \\ r_{ij} - r_{k+1,j}, & \text{for } i \geq k+1 \end{cases}$$

Then $q_j \geq 0$, $t_i \geq 0$ for every i, j and $q_j > 0$ if $j \neq 1$ and $t_i > 0$ if $i \neq k+1$.

Let $A_i = r_{i1} \sum_j w_{ij}$. Since $\sum_j r_{ij} w_{ij} = 0$ for every i by the equilibrium condition at vertices in X, we have

$$\begin{aligned} A_i &= r_{i1} \sum_j w_{ij} - \sum_j r_{ij} w_{ij} \\ &= \sum_j (r_{i1} - r_{ij}) w_{ij} \\ &= \begin{cases} -\sum_j q_j w_{ij}, & \text{for } i \leq k \\ \sum_j q_j w_{ij}, & \text{for } i \geq k+1 \end{cases} \end{aligned}$$

Let $B_j = r_{k+1,j} \sum_i w_{ij}$. Since $\sum_{i \leq k} r_{ij} w_{ij} - \sum_{i \geq k+1} r_{ij} w_{ij} = 0$ for every $j = 1, \ldots, n$ by the equilibrium condition at vertices in Y, we have

$$\begin{aligned} B_j &= \sum_{i \leq k} (r_{k+1,j} + r_{ij}) w_{ij} + \sum_{i \geq k+1} (r_{k+1,j} - r_{ij}) w_{ij} \\ &= \sum_{i \leq k} t_i w_{ij} - \sum_{i \geq k+1} t_i w_{ij}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_i t_i A_i &= \sum_{i\le k} t_i A_i + \sum_{i\ge k+1} t_i A_i \\
&= \sum_{i\le k} t_i(-\sum_j q_j w_{ij}) + \sum_{i\ge k+1} t_i(\sum_j q_j w_{ij}) \\
&= -\sum_j q_j \sum_{i\le k} t_i w_{ij} + \sum_j q_j \sum_{i\ge k+1} t_i w_{ij} \\
&= -\sum_j q_j B_j.
\end{aligned}$$

Since Ω is PSD and $r_{ij} > 0$, $A_i, B_j \ge 0$ hold for every i, j. Hence $0 \le \sum_i t_i A_i = -\sum_j q_j B_j \le 0$ holds. Therefore, equality must occur everywhere, which means that $A_i = 0$ for $i \ne k+1$ and $B_j = 0$ for $j \ne 1$. Thus every entry on the diagonal of Ω with possible exceptions of the entries corresponding to (x_{k+1}, x_{k+1}) and (y_1, y_1) must be zero. However, by using the symmetry of the graph, we can deduce that these entries must also be zero, so every entry on the diagonal of the PSD matrix Ω is zero. Therefore, Ω is the zero matrix. Theorem 8.2.2 now implies that $(G, \mathbf{p})$ is not universally rigid; in fact, it has an equivalent realization of affine dimension $m + n - 1$.

3. If none of the conditions in 1 and 2 holds, then there exist, say x_1, x_2, y_1, y_2, such that $p(x_1) < p(y_1) < p(x_2) < p(y_2)$. Then x_1, y_1, x_2, y_2 form a stretched cycle in $(G, \mathbf{p})$ and hence x_1, x_2 and y_1, y_2 are universally linked in $(G, \mathbf{p})$. This implies that the pairwise distances among these four vertices are the same in all realizations of G equivalent to $(G, \mathbf{p})$ and hence $(G, \mathbf{p})$ is universally rigid if and only if $(G', \mathbf{p})$ is universally rigid, where $G' = G + x_1x_2 + y_1y_2$. It remains to observe that $(G', \mathbf{p})$ can be obtained from a framework on a complete graph on four vertices by iteratively attaching vertices of degree two (and adding edges). These operations are known to preserve universal rigidity on the line (Lemma 3.3.10). Therefore $(G', \mathbf{p})$ and hence $(G, \mathbf{p})$ are universally rigid, as required. □

Theorem 8.2.1 implies the following observation of Connelly.

Corollary 8.2.3 (Connelly [30])**.** *The only generically universally rigid bipartite graph in* $\mathbb{R}^1$ *is the single edge* $K_{1,1}$.

8.2.2 Bipartite graphs in general dimension

The proof of Theorem 8.2.1 implies the following result in general dimension.

Theorem 8.2.4. *The only generically universally rigid bipartite graph in* $\mathbb{R}^d$ *is the single edge* $K_{1,1}$*, for every dimension* $d \geq 1$*.*

Proof. First we observe that no general position d-dimensional realization of a non-complete graph on at most $d+1$ vertices is locally rigid. Thus the only d-dimensional generically universally rigid bipartite graph on at most $d+1$ vertices is $K_{1,1}$. Next consider a complete bipartite graph G on at least $d+2$ vertices and a d-dimensional generic realization $(G, \mathbf{p})$ with the property that the projected one-dimensional framework $(G, \mathbf{p}')$, obtained by projecting the configuration $\mathbf{p}$ to one coordinate axis, satisfies Condition 1 or 2 in Theorem 8.2.1 (Figure 8.3).

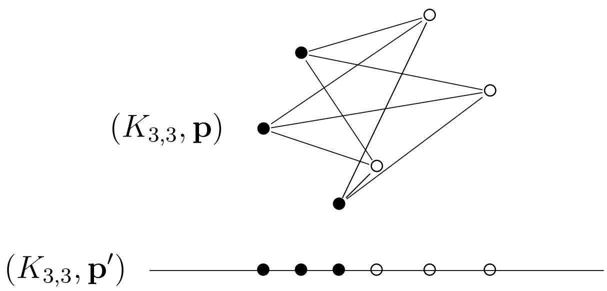


Figure 8.3: A generic complete bipartite framework in $\mathbb{R}^2$ and its projection on a line.

A PSD stress matrix of $(G, \mathbf{p})$ is also a PSD stress matrix of $(G, \mathbf{p}')$ and hence, by the proof of Theorem 8.2.1, it must be the zero matrix. Thus, by Theorem 8.2.2, $(G, \mathbf{p})$ is not universally rigid. Therefore, a complete bipartite graph, except the single edge $K_{1,1}$, is not generically universally rigid in any dimension.

□

This result contrasts universal rigidity with local and global rigidity. While local and global rigidity are known to be implied by a high connectivity in dimension $1, 2$ (and conjectured to be so in higher dimensions), Theorem 8.2.4 shows that high connectivity does not imply universal rigidity.

8.2.3 Observations, open questions, conjectures

The study of the universal rigidity of bipartite frameworks gives rise to many open questions and conjectures. In the remain of this section, we introduce and discuss these questions and conjectures, whose answers would provide deeper understanding of universal rigidity and could lead to combinatorial characterizations of universal rigidity at least in low dimensions.

The first question is motivated by Theorem 8.2.2.

Question 1. *Is it true that a generic framework* $(G, \mathbf{p})$ *has a PSD stress matrix* Ω *of rank at least* k *if and only if* $(G, \mathbf{p})$ *has no equivalent realization of affine dimension* $|V| - i$ *for any* i *where* $1 \leq i \leq k$*.*

Note that when $k = 1$ we have Theorem 8.2.2. The answer is "no" if we replace generic position by general position. (See the example in Section 8.3 Figure 8.10(b).) The "only if" part follows from the following result.

Theorem 8.2.5 (Alfakih [4])**.** *Let* $(G, \mathbf{p})$ *be a framework and* Ω *a PSD stress matrix of* $(G, \mathbf{p})$*. Then* Ω *is a stress matrix for every framework* $(G, \mathbf{q})$ *equivalent to* $(G, \mathbf{p})$*.*

(cf. Theorem 8.3.4)

In fact, suppose that Ω is a PSD stress matrix of $(G, \mathbf{p})$ of rank at least k, and $(G, \mathbf{q})$ a framework equivalent to $(G, \mathbf{p})$. Then Ω is a stress matrix for $(G, \mathbf{q})$. If d is the affine dimension of $(G, \mathbf{q})$ then $\operatorname{rank} \Omega \leq |V| - d - 1$. Therefore, $d \leq |V| - k - 1$.

Theorem 8.2.1 also leads us to the following question.

Question 2. *Is it true that the universal rigidity of a general position framework* $(G, \mathbf{p})$ *in* $\mathbb{R}^1$ *depends only on the ordering of vertices on the line (and not on the coordinates)?*

As remarked in the previous section, a highly connected complete bipartite graph is not generically universally rigid in $\mathbb{R}^1$ (1-GUR). However, if we add one edge to this graph, it becomes 1-GUR and many edges are redundant (Figure 8.4). This observation inspires the following question on minimally 1-GUR graphs, i.e., 1-GUR graph that is no more 1-GUR after the deletion of any edge.

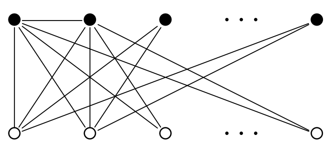


Figure 8.4: A sparse 1-GUR subgraph of a complete bipartite graph plus one edge.

Question 3. *Let* $G = (V, E)$ *be a minimally* 1*-GUR graph. Does there exists a (linear) upper bound on* $|E|$ *as a function of* $|V|$*?*

Now let us recall the Ratmanski's construction in Lemma 3.3.10: if we glue two 1-GUR graphs over at least 2 vertices then we obtain another 1-GUR graph. In order to prove the conjecture that the converse is also true, one should study the effect of cleavage on a 1-GUR graph. Let $G = (V, E)$ be a graph. A pair (G_1, G_2), where G_1, G_2 are subgraphs of G, is called a k-separator of G if $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, and $|V(G_1) \cap V(G_2)| = k$ hold. For a subset $X \subseteq V$ let $G + K(X)$ denote the supergraph of G obtained by adding all edges connecting pairs of vertices of X (which are non-adjacent in G). We have the following simple result.

Lemma 8.2.6. *Let G be a 1-GUR graph and (G_1, G_2) a k-separator of G with $X = V(G_1) \cap V(G_2)$. Then $G_i + K(X)$ is 1-GUR for $i = 1, 2$.*

Proof. Suppose that $\hat{G}_1 = G_1 + K(X)$ is not 1-GUR. Then there exists a generic realization $(\hat{G}_1, \mathbf{p}_1)$ of $\hat{G}_1$ in $\mathbb{R}^1$ which is not univerally rigid and hence there exists a realization $(\hat{G}_1, \mathbf{p}'_1)$ equivalent but non congruent to $(\hat{G}_1, \mathbf{p}_1)$. We can assume that $p'_1(v) = p_1(v)$ for every v in X. Extend $\mathbf{p}_1$ to a generic realization $\mathbf{p}$ of G in $\mathbb{R}^1$. Let

$$p'(v) = \begin{cases} p'_1(v), & v \in V(G_1) \\ p(v), & v \in V(G_2) \end{cases}$$

Then $(G, \mathbf{p}')$ is equivalent but not congruent to $(G, \mathbf{p})$, which means that G is not 1-GUR, a contradiction. □

In particular, a 1-GUR graph can be cut into two 1-GUR graphs along a cut pair u, v if we add the edge uv into both sides. The addition of this edge uv is undesirable when uv is not originally presented in G. What one may expect is that one of the two graphs in the 2-separator is 1-GUR.

Conjecture 8.2.7. *Let G be 1-GUR and (G_1, G_2) a 2-separation in G with $V(G_1) \cap V(G_2) = \{x, y\}$. Then G_1 or G_2 is 1-GUR.*

The following conjecture might be helpful in proving this. A pair of vertices $\{u, v\}$ in G is called universally linked in $\mathbb{R}^d$ if $\{u, v\}$ is universally linked in all d-dimensional generic realizations $(G, \mathbf{p})$ of G.

Conjecture 8.2.8. *Suppose that a pair $\{u, v\}$ is not universally linked in G in $\mathbb{R}^1$. Then there exist generic 1-dimensional realizations $(G, \mathbf{p}), (G, \mathbf{q})$ of G and a realization $(G, \mathbf{p}')$ equivalent to $(G, \mathbf{p})$, a realization $(G, \mathbf{q}')$ equivalent to $(G, \mathbf{q})$, such that $\|p'(u) - p'(v)\| > \|p(u) - p(v)\|$ and $\|q'(u) - q'(v)\| < \|q(u) - q(v)\|$.*

By assuming the truth of Conjecture 8.2.8 we can show that if G_1 and G_2 are not 1-GUR then there exists a general position realization $(G, \mathbf{p})$ on the line which is not universally rigid. We believe that $\mathbf{p}$ can be made generic, too (c.f. Question 2), which would imply Conjecture 8.2.7.

Our argument is as follows. We may assume that $\{x, y\}$ is not universally linked in G_1 and G_2. Thus there is a generic realization $(G_1, \mathbf{p})$ in $\mathbb{R}^1$ and an equivalent realization $(G_1, \mathbf{q})$ such that the distance between $p(x)$ and $p(y)$ is, say, stricly smaller than the distance between $q(x)$ and $q(y)$. By assuming the truth of Conjecture 8.2.8 we can find a generic realization $(G_2, \mathbf{p}')$ in $\mathbb{R}^1$ and an equivalent realization $(G_2, \mathbf{q}')$ such that the distance between $p'(x)$ and $p'(y)$ is, say, stricly smaller than the distance between $q'(x)$ and $q'(y)$. By carefully choosing the generic realization $(G_2, \mathbf{p}')$ and rescaling, if necessary, we may assume that $\|p(x)-p(y)\| = \|p'(x)-p'(y)\|$. Now using Lemma 3.2.5 we can obtain a pair of realizations $(G_1, \mathbf{r})$ and $(G_2, \mathbf{r}')$ for which $\|r(x)-r(y)\| = \|r'(x)-r'(y)\| > \|p(x)-p(y)\|$ and such that $(G_1, \mathbf{r})$ is equivalent to $(G_1, \mathbf{p})$ and $(G_2, \mathbf{r}')$ is equivalent to $(G_2, \mathbf{p}')$. By glueing together $(G_1, \mathbf{p})$ and $(G_2, \mathbf{p}')$ as well as $(G_1, \mathbf{r})$ and $(G_2, \mathbf{r}')$ along the pair x, y we obtain two equivalent but not congruent realizations of G, where the former realization is 1-dimensional. The configuration obtained by glueing $(G_1, \mathbf{p})$ and $(G_2, \mathbf{p}')$ together may not be generic. However, we believe that a generic non 1-UR framework can be obtained if we perturb it to a close enough generic one on the line.

Another possible approach to the difficult problem of characterizing universal rigidity is study classes of non generically universally rigid (non GUR) graphs. The proof of Theorem 8.2.1 shows that a bipartite framework on the line is not universally rigid when there is no stretched cycle. Therefore, the class of graphs which have a realization on the line without stretched cycles is an interesting candidate for non 1-GUR graphs. In fact, this class of graphs is known as *cover graphs* and they can be defined in the following manner. Let $G = (V, E)$ be a graph and let $\vec{G}$ be an acyclic orientation of G. An edge e of G is *dependent* if the reversal of e in $\vec{G}$ creates a directed cycle. An orientation without dependent edges is called *strongly acyclic*. We say that G is a *cover graph* if G has a strongly acyclic orientation. (It is known that G is a cover graph if and only if it is the Hasse diagram of some partially ordered set on V.) One can see that a cover graph can be realized as a framework on the line without stretched cycles: we can order the vertices of a cover graph on the line so that if uv is an arc then u is on the right of v. Conversely, a realization as a framework on the line without stretched cycles of a graph naturally gives a strongly acyclic orientation of the graph by orienting every edges from left to right. Furthermore, cover graphs are triangle-free. Ratmanski's construction adds more reason to think that the following question has an affirmative answer.

Question 4. *Is it true that no cover graph is* 1-*GUR (except* $K_{1,1}$*)?*

We should also remark that it is NP-hard to test whether a given graph is a cover graph [17, 84].

Some special subclasses of cover graphs may also be interesting. One of them is triangle-free 3-colorable graphs, for which a realization on the line without stretched cycles looks quite similar to the second configuration in Theorem 8.2.1. These graphs are known to be cover graphs, a fact followed from a more general result that every k-colorable graph without cycles of length at most k is a cover graph [34]. Also, one may first check the question for triangle-free planar graphs.

Question 5. *Is it true that no triangle-free planar graph (or even triangle-free* 3-*colorable graph) is* 1-*GUR (except* $K_{1,1}$*)?*

We may also ask whether all non cover graphs are 1-GUR. An interesting graph to analyze is the Grötzsch graph, which is triangle-free and 4-chromatic, see Figure 8.5. This graph is not a cover graph [34]. Is it 1-GUR? Since it is triangle-free, an affirmative answer to this question will imply that Ratmanski's construction is not sufficient to generate all 1-GUR graphs.

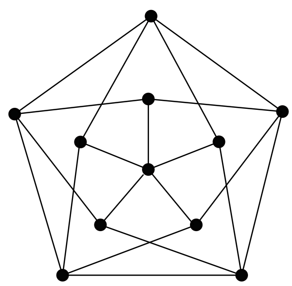


Figure 8.5: The Grötzsch graph.

Sparse graphs and cover graphs

The rest of this section contains some further questions and observations about the relation between sparse graphs and cover graphs. Let $G = (V, E)$ be a simple graph. We recall that G is $(2, 4)$-*sparse* if for all subsets $X \subseteq V$ with $|X| \geq 3$ the subgraph induced by X has at most $2|X| - 4$ edges. For example, triangle-free planar graphs are $(2,4)$-sparse.

Question 6. *Is every* $(2,4)$*-sparse graph a cover graph?*

A $(2,4)$-sparse graph is clearly triangle-free. It is also independent in the 2-dimensional generic rigidity matroid by Laman's theorem. This leads us to a further extension:

Question 7. *Is every triangle-free graph which is independent in the* 2*-dimensional generic rigidity matroid a cover graph?*

One proof method for a positive result here would use the Henneberg operations. The next lemmas on the construction of cover graphs show that this approach may be useful.

Lemma 8.2.9. *Let G be a triangle-free graph obtained from a graph H by a* 0*-extension operation. Then G is a cover graph if and only if H is a cover graph.*

Proof. Since H is a subgraph of G, necessity is obvious. To see the other direction consider a strongly acyclic orientation $\mathcal{H}$ of H. Suppose that $G = H + vx + vy$.

Since $\mathcal{H}$ is acyclic, we cannot have an (x, y)-directed path and a (y, x)-directed path in $\mathcal{H}$ simultaneously. Thus we have three cases to consider.

<u>Case 1</u>: There is an (x, y)-directed path in $\mathcal{H}$. Then we orient vx from x to v and vy from v to y.

<u>Case 2</u>: There is an (y, x)-directed path in $\mathcal{H}$. Then we orient vx from v to x and vy from y to v.

<u>Case 3</u>: There is neither (x, y)-directed path nor (y, x)-directed path in $\mathcal{H}$. Then we orient vx from v to x and vy from v to y.

Lemma 8.2.10. *Let G be a triangle-free graph obtained from a cover graph H by a* 1*-extension operation. Then G is also a cover graph.*

Proof. Consider a strongly acyclic orientation $\mathcal{H}$ of H. Suppose that $G = H - xy + vx + vy + vz$. We orient the edges vx, vy, vz as follows.

<u>Case 1</u>: There is an (x, z)-directed path in $\mathcal{H} - xy$. Then there is no (z, y)-directed path in $\mathcal{H} - xy$.
<u>Case 1.1</u>: There is a (y, z)-directed path in $\mathcal{H} - xy$. We orient vx from x to v, vy from y to v and vz from v to z. (Figure 8.6)

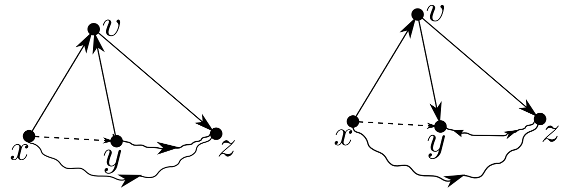


Figure 8.6: Cases 1.1 and 1.2

<u>Case 1.2</u>: There is no (y, z)-directed path in $\mathcal{H} - xy$. We orient vx from x to v, vy from v to y and vz from v to z. (Figure 8.6)

<u>Case 2</u>: There is an (z, x)-directed path in $\mathcal{H} - xy$. Then every (y, z)-path in $\mathcal{H} - xy$ has at least two backward edges. We orient vx from v to x, vy from v to y and vz from z to v. (Figure 8.7)

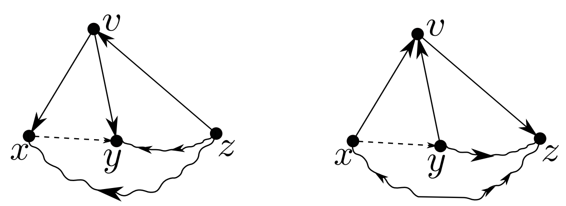


Figure 8.7: Case 2 and Case 3.1

<u>Case 3</u>: There is neither (x, z)-directed path nor (z, x)-directed path in $\mathcal{H} - xy$.
<u>Case 3.1</u>: There is a (y, z)-directed path in $\mathcal{H} - xy$. Then every (x, z)-path in $\mathcal{H} - xy$ has at least two forward edges. We orient vx from x to v, vy from y to v and vz from v to z. (Figure 8.7)
<u>Case 3.2</u>: There is a (z, y)-directed path in $\mathcal{H} - xy$. We orient vx from x to v, vy from v to y and vz from z to v. (Figure 8.8)

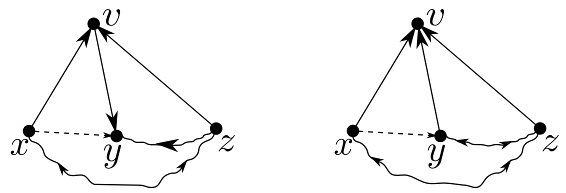


Figure 8.8: Cases 3.2 and 3.3

<u>Case 3.3</u>: There is neither (y, z)-directed path nor (z, y)-directed path in $\mathcal{H} - xy$. We orient vx from x to v, vy from y to v and vz from z to v. (Figure 8.8) $\square$

8.3 Universal rigidity of tensegrity frameworks

8.3.1 Introduction

In this section, we make a convention that

all vectors (or points) are considered to be column vectors.

An r-dimensional *tensegrity framework* is a pair $(G, \mathbf{p})$ where $G = (V, E)$ is a graph whose edges are labelled as a bar, a cable or a strut and $\mathbf{p}$ is an assignment of a point $p(v)$ in $\mathbb{R}^r$ to each vertex $v \in V$. For ease of notation, in this section, we will identify the vertex set V with $\{1, \ldots, n\}$ where $n = |V|$ and write p^i instead of $p(i)$. The *affine dimension* of the configuration $\mathbf{p}$ or the framework $(G, \mathbf{p})$, is the dimension of the affine space spanned by $p^1, \ldots, p^n$. If the affine dimension of $\mathbf{p}$ is equal to the embedding dimension r then we say that $(G, \mathbf{p})$ is of *full dimension*.

Let B, C and S denote the sets of bars, cables and struts of $(G, \mathbf{p})$ respectively. Thus

$$E = B \cup C \cup S.$$

A *bar framework* (which is in fact our familiar bar-joint framework) $(G, \mathbf{p})$ is a tensegrity framework where $E = B$, i.e., $C = S = \emptyset$.

Let $(G, \mathbf{p})$ and $(G, \mathbf{q})$ be two tensegrity frameworks in $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. $(G, \mathbf{q})$ is said to be *congruent* to $(G, \mathbf{p})$ if

$$\|q^i - q^j\|^2 = \|p^i - p^j\|^2 \quad \text{for all } i, j = 1, \ldots, n.$$

On the other hand, we say that $(G, \mathbf{q})$ is *dominated* by $(G, \mathbf{p})$, (or $(G, \mathbf{p})$ *dominates* $(G, \mathbf{q})$), if

$$\begin{array}{rcll} \|q^i - q^j\|^2 & = & \|p^i - p^j\|^2 & \text{for each } \{i, j\} \in B, \\ \|q^i - q^j\|^2 & \leq & \|p^i - p^j\|^2 & \text{for each } \{i, j\} \in C, \\ \|q^i - q^j\|^2 & \geq & \|p^i - p^j\|^2 & \text{for each } \{i, j\} \in S. \end{array} \tag{8.3.1}$$

A tensegrity framework $(G, \mathbf{q})$ in $\mathbb{R}^r$ is said to be *affinely-dominated* by $(G, \mathbf{p})$ if $(G, \mathbf{q})$ is dominated by $(G, \mathbf{p})$ and $q^i = Ap^i + b$ for all $i = 1, \ldots, n$, where A is an $r \times r$ matrix and $b \in \mathbb{R}^r$.

A tensegrity framework $(G, \mathbf{p})$ in $\mathbb{R}^r$ is said to be *dimensionally rigid* if no tensegrity framework $(G, \mathbf{q})$ in any dimension $\geq r + 1$ is dominated by $(G, \mathbf{p})$. We say that a tensegrity framework $(G, \mathbf{p})$ is *universally rigid* if every tensegrity framework $(G, \mathbf{q})$ in any dimension that is dominated by $(G, \mathbf{p})$ is in fact congruent to $(G, \mathbf{p})$.

Let $(G, \mathbf{p})$ be a tensegrity framework on n vertices in $\mathbb{R}^r$. An *equilibrium stress* of $(G, \mathbf{p})$ is a real valued function ω on E such that

$$\sum_{j:\{i,j\}\in E} \omega_{ij}(p^i - p^j) = \mathbf{0} \text{ for all } i = 1, \ldots, n. \tag{8.3.2}$$

An equilibrium stress ω is said to be *proper* if $\omega_{ij} \geq 0$ for all $\{i, j\} \in C$ and $\omega_{ij} \leq 0$ for all $\{i, j\} \in S$. Let $\overline{E}$ denote the set of missing edges in G, i.e.,

$$\overline{E} = \{\{i, j\} : i \neq j, \{i, j\} \notin E\},$$

and let ω be an equilibrium stress of $(G, \mathbf{p})$. Then the $n \times n$ symmetric matrix Ω where

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{if } \{i,j\} \in E, \\ 0 & \text{if } \{i,j\} \in \overline{E}, \\ \displaystyle\sum_{k:\{i,k\}\in E} \omega_{ik} & \text{if } i = j, \end{cases} \tag{8.3.3}$$

is called the *stress matrix* associated with ω, or a stress matrix of $(G, \mathbf{p})$. The stress matrix Ω associated with a proper equilibrium ω is called *proper*.

The following result provides a sufficient condition for the universal rigidity of a given tensegrity framework.

Theorem 8.3.1 (see [76, 21, 23])**.** *Let $(G, \mathbf{p})$ be a tensegrity framework of full dimension in $\mathbb{R}^r$, for some $r \leq n - 2$. Suppose that the following two conditions hold.*

1. *There exists a proper PSD stress matrix Ω of $(G, \mathbf{p})$ of rank $n - r - 1$.*
2. *There does not exist a tensegrity framework $(G, \mathbf{q})$ in $\mathbb{R}^r$ that is affinely-dominated by, but not congruent to, $(G, \mathbf{p})$.*

Then $(G, \mathbf{p})$ is universally rigid.

Note that $(n - r - 1)$ is the maximum possible value for the rank of the stress matrix Ω. As we will see in the next, the existence of a proper PSD stress matrix of maximal rank in condition 1 implies that every framework dominated by $(G, \mathbf{p})$ is obtained from $(G, \mathbf{p})$ by an affine transformation. Condition 2 excludes those frameworks that are not congruent to $(G, \mathbf{p})$.

When $(G, \mathbf{p})$ is a generic framework, condition 2 in Theorem 8.3.1 is implied by condition 1.

Theorem 8.3.2 ([21, 23, 76])**.** *Let $(G, \mathbf{p})$ be a generic tensegrity framework of full dimension in $\mathbb{R}^r$ for some $r \leq n-2$. If $(G, \mathbf{p})$ has a proper PSD stress matrix Ω with $\operatorname{rank} \Omega = n - r - 1$ then $(G, \mathbf{p})$ is universally rigid.*

Alfakih and Ye showed that the condition on genericity in the previous theorem can be weakened for bar frameworks. A configuration $\mathbf{p}$ is said to be in *general position* if every subset of $\{p^1, \ldots, p^n\}$ of cardinality at most $r+1$ is affinely independent.

Theorem 8.3.3 (Alfakih and Ye [8])**.** *Let $(G, \mathbf{p})$ be a bar framework on n vertices in general position in $\mathbb{R}^r$ with $r \leq n-2$. If $(G, \mathbf{p})$ has a proper PSD stress matrix Ω with $\operatorname{rank} \Omega = n - r - 1$ then $(G, \mathbf{p})$ is universally rigid.*

Contribution: In this section we first extend these sufficient conditions to tensegrity frameworks in general position.

Theorem 8.3.4. *Let $(G, \mathbf{p})$ be a tensegrity framework on n vertices in general position in $\mathbb{R}^r$ for some $r \leq n-2$. Suppose that $(G, \mathbf{p})$ is of full dimension. If $(G, \mathbf{p})$ has a PSD stress matrix Ω with $\operatorname{rank} \Omega = n-r-1$ then $(G, \mathbf{p})$ is universally rigid.*

For bar frameworks, the result can be furthermore strengthened. For a vertex i of $\{1, \ldots, n\}$, let $N(i)$ denote the set of neighbors of i, that is, the set of vertices adjacent to i.

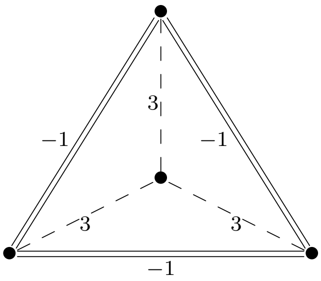
Theorem 8.3.5. *Let $(G, \mathbf{p})$ be a bar framework on n vertices in $\mathbb{R}^r$ for some $r \leq n-2$. Suppose that the following conditions hold.*

1. *$(G, \mathbf{p})$ has a PSD stress matrix Ω with $\operatorname{rank} \Omega = n - r - 1$*
2. *For each vertex $i \in \{1, \ldots, n\}$, the set $\{p^j : j \in \{i\} \cup N(i)\}$ affinely spans $\mathbb{R}^r$.*

Then $(G, \mathbf{p})$ is universally rigid.

The proofs of these results will be given in Section 8.3.6. To this end we characterize dominated tensegrity frameworks and affinely dominated tensegrity frameworks in Section 8.3.4 and 8.3.5 respectively. Lastly, Section 8.3.7 discusses an example showing that the converse of Theorem 8.3.4 is not true.

The results in this section are from a joint work with Abdo Alfakih [8].



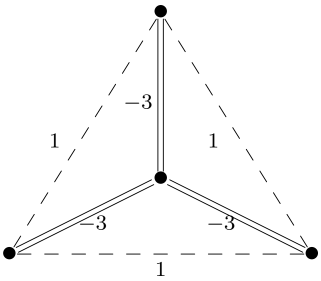


Figure 8.9: Two tensegrity frameworks. Cables are denoted by dashed lines and struts by double lines.
(a) The framework is universally rigid. The associated proper stress matrix is PSD of rank $1 = 4 - 2 - 1$.
(b) The framework is not universally rigid. The only proper stress matrix (up to positive multiple) is negative semidefinite.

Example: Consider two tensegrity frameworks in the plane in Figure 8.9 where the outer triangles are regular and the fourth vertices are in the center. We denote cables by dashed lines and struts by double lines. The unique proper stress (up to a positive scalar multiple) is indicated on the figure in each case. It is easy to see that the framework in (a) is universally rigid while the framework in (b) is not. The corresponding proper stress matrices are

$$\Omega_1 = \begin{pmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & -3 \\ -3 & -3 & -3 & 9 \end{pmatrix} \quad \text{and} \quad \Omega_2 = -\Omega_1.$$

Ω_1 is PSD and $\operatorname{rank}\Omega_1 = 1 = 4 - 2 - 1$ while Ω_2 is negative semidefinite.

8.3.2 Preliminaries

Let $\mathbf{e}^i$ denote the ith unit vector in $\mathbb{R}^n$ and let $\mathbf{1}$ denote the vector of all 1's in $\mathbb{R}^n$. Define the following $n \times n$ symmetric matrices where $i < j$.

$$\begin{aligned} F^{ij} &= (\mathbf{e}^i - \mathbf{e}^j)(\mathbf{e}^i - \mathbf{e}^j)^T, \\ E^{ij} &= \mathbf{e}^i(\mathbf{e}^j)^T + \mathbf{e}^j(\mathbf{e}^i)^T, \\ L^i &= \mathbf{e}^i\mathbf{1}^T + \mathbf{1}(\mathbf{e}^i)^T. \end{aligned} \tag{8.3.4}$$

Recall the Kronecker delta δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following two lemmas easily follow.

Lemma 8.3.6. *We have the following equalities.*

$$\langle F^{kl}, E^{ij} \rangle = -2\delta_{ki}\delta_{lj} \tag{8.3.5}$$

for $1 \leq k < l \leq n, 1 \leq i < j \leq n$, *and*

$$\langle F^{kl}, L^{i} \rangle = 0, \tag{8.3.6}$$

for $1 \leq k < l \leq n$ *and* $1 \leq i \leq n$.

Equation (8.3.5) implies that F^{kl} is orthogonal to E^{ij} if $(k,l) \neq (i,j)$ and equation (8.3.6) implies that F^{kl} is orthogonal to L^i for every $1 \leq k < l \leq n$ and $1 \leq i \leq n$.

Lemma 8.3.7.

1. *The set* $\{F^{ij} : 1 \leq i < j \leq n\}$ *is linearly independent.*

2. *The set* $\{E^{ij} : 1 \leq i < j \leq n\} \cup \{L^i : 1 \leq i \leq n\}$ *is linearly independent.*

An immediate consequence of Lemmas 8.3.6 and 8.3.7 is as follows.

Corollary 8.3.8. *Let* N *be a subset of* $\{(i,j) : 1 \leq i < j \leq n\}$ *and* $\mathcal{L}$ *the linear space spanned by* $\{F^{ij} : (i,j) \in N\}$. *Then the linear space spanned by* $\{E^{ij} : 1 \leq i < j \leq n, (i,j) \notin N\} \cup \{L^i : 1 \leq i \leq n\}$ *is the orthogonal space* $\mathcal{L}^{\perp}$ *of* $\mathcal{L}$ *in* $\mathcal{S}_n$.

Proof. By Lemma 8.3.6, the linear subspace spanned by $\{E^{ij} : 1 \leq i < j \leq n, (i,j) \notin N\} \cup \{L^i : 1 \leq i \leq n\}$ is orthogonal to the linear space $\mathcal{L}$ spanned by $\{F^{ij} : (i,j) \in N\}$. Moreover, by Lemma 8.3.7, $\{F^{ij} : (i,j) \in N\}$ is linearly independent hence a base of $\mathcal{L}$. Also, $\{E^{ij} : 1 \leq i < j \leq n\} \cup \{L^i : 1 \leq i \leq n\}$ is linearly independent. A simple count shows that the co-dimension of $\mathcal{L}$ is equal to the cardinality of $\{E^{ij} : 1 \leq i < j \leq n\} \cup \{L^i : 1 \leq i \leq n\}$. The result follows. $\square$

8.3.3 The configuration matrix and Gale matrices

The *configuration matrix* P of $\mathbf{p}$ is the $n \times r$ matrix whose i-th row is $(p^i)^T$, i.e.,

$$P = \begin{pmatrix} (p^1)^T \\ \vdots \\ (p^n)^T \end{pmatrix}$$

Then we have

$$\langle F^{ij}, PP^T \rangle = \|p^i - p^j\|^2 \quad \text{for } 1 \le i < j \le n, \tag{8.3.7}$$

since $\langle F^{ij}, PP^T \rangle = \operatorname{trace}(F^{ij}PP^T) = \operatorname{trace}((\mathbf{e}^i - \mathbf{e}^j)(\mathbf{e}^i - \mathbf{e}^j)^T PP^T) = (\mathbf{e}^i - \mathbf{e}^j)^T PP^T(\mathbf{e}^i - \mathbf{e}^j) = (p^i - p^j)(p^i - p^j)^T = \|p^i - p^j\|^2$.

The *augmented configuration matrix* $\hat{P}$ is the $n \times (r+1)$ matrix obtained from P by adding an all-one column vector to the right of P. Then an embedding $\mathbf{p}$ in $\mathbb{R}^n$ is in general position if and only if every $r+1$ rows of $\hat{P}$ are linearly independent. It is easy to see that if Ω is a stress matrix for $(G, \mathbf{p})$ then $\Omega\hat{P} = \mathbf{0}$. Therefore we have the following simple but important consequence.

Lemma 8.3.9. *If Ω is a stress matrix for a framework $(G, \mathbf{p})$ then the affine dimension of $\mathbf{p}$ is at most $n - 1 - \operatorname{rank}\Omega$.*

Proof. The affine dimension of $\mathbf{p}$ is $\operatorname{affdim}(\mathbf{p}) = \operatorname{rank}\hat{P} - 1$. Since $\Omega\hat{P} = 0$ we have $\operatorname{rank}\Omega \le n - \operatorname{rank}\hat{P} \le n - \operatorname{affdim}(\mathbf{p}) - 1$. Therefore $\operatorname{affdim}(\mathbf{p}) \le n - 1 - \operatorname{rank}\Omega$. ☐

Suppose that $\mathbf{p}$ is a configuration of $\{1, \dots, n\}$ in $\mathbb{R}^r$ that affinely spans $\mathbb{R}^r$. Let $\bar{r}$ denote $n - r - 1$. A *Gale matrix* of $\mathbf{p}$ is an $n \times \bar{r}$ matrix Z whose columns form a base of the nullspace of $\hat{P}^T$. Let $(z^i)^T$ be the i-th row of Z. The vector z^i is called the *Gale transformation* of p^i, for $i = 1, \dots, n$.

The following lemma is easy but useful.

Lemma 8.3.10. *If Z is a Gale matrix of a configuration $\mathbf{p}$ and Q is any non-singular $\bar{r} \times \bar{r}$ matrix then $Z' = ZQ$ is also a Gale matrix of $\mathbf{p}$. On the other hand, if Z, Z' are Gale matrices of $\mathbf{p}$ then there exists a non-singular $\bar{r} \times \bar{r}$ matrix Q such that $Z' = ZQ$.*

Proof. If Z is a Gale matrix of a configuration $\mathbf{p}$ and $Z' = ZQ$ for a non-singular $\bar{r} \times \bar{r}$ matrix Q then the columns of Z' are linearly independent. Moreover, $\hat{P}^T Z' = \hat{P}^T ZQ = 0$. Hence, Z' is a Gale matrix of $\mathbf{p}$.

Conversely, if Z, Z' are Gale matrices of $\mathbf{p}$ then the columns of Z and the columns of Z' form bases of the same linear space. Therefore, $Z' = ZQ$ for some non-singular $\bar{r} \times \bar{r}$ matrix Q. ☐

For a matrix M and a subset J of the row index set of M, we denote by M_J the submatrix of M whose rows are indexed by J. The following observation on the relation between the augmented configuration matrix and Gale matrix is crucial in the proof of our main theorems.

Lemma 8.3.11. *Let $(G, \mathbf{p})$ be a framework in $\mathbb{R}^r$ and Z a Gale matrix of $\mathbf{p}$. Suppose that $J \subseteq V$ satisfies $|J| = r + 1$ and $\hat{P}_J$ is non-singular. Then $Z_{\bar{J}}$ is non-singular, where $\bar{J} = V \setminus J$.*

Proof. Suppose that λ_i for $i \in \bar{J}$ are scalars satisfying $\sum_{i \in \bar{J}} \lambda_i z^i = 0$. We show that $\lambda_i = 0$ for all $i \in \bar{J}$.

Set $\lambda_i = 0$ for $i \in J$ and $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, we have $Z^T \lambda = \mathbf{0}$. Since the columns of $\hat{P}$ form a base of the nullspace of Z^T, we deduce that $\lambda = \hat{P}x$ for some $x \in \mathbb{R}^{r+1}$. Then, $\mathbf{0} = \lambda_J = \hat{P}_J x$ holds. The non-singularity of $\hat{P}_J$ implies that $x = \mathbf{0}$. Thus $\lambda = \mathbf{0}$. This completes the proof. $\square$

Corollary 8.3.12. *Let $(G, \mathbf{p})$ be a framework in $\mathbb{R}^r$ and Z a Gale matrix of $\mathbf{p}$. Suppose that $J \subseteq V$ satisfies that the set $\{p^i : i \in J\}$ affinely spans $\mathbb{R}^r$. Then the rows of the matrix $Z_{\bar{J}}$ are linearly independent, where $\bar{J} = V \setminus J$.*

Proof. Let $J' \subseteq J$ satisfy $|J'| = r + 1$ and $\{p^i : i \in J'\}$ is affinely independent. Then Lemma 8.3.11 applied to J' implies that $Z_{\bar{J'}}$ is non-singular. Since $J' \subseteq J$, the rows of $Z_{\bar{J}}$ are also rows of $Z_{\bar{J'}}$ so the corollary follows. $\square$

8.3.4 Dominated tensegrity frameworks

Without loss of generality we assume that

Assumption 8.3.1. *For every configuration $\mathbf{p}$, the centroid of the points $p^1, \ldots, p^n$ coincides with the origin, i.e., $P^T \mathbf{1} = \mathbf{0}$.*

For ease of notation, we define

$$\mathcal{E}(y) = \sum_{\{i,j\} \in \overline{E} \cup C \cup S} y_{ij} E^{ij}, \tag{8.3.8}$$

where $y = (y_{ij}) \in \mathbb{R}^{|\overline{E}|+|C|+|S|}$.

The next lemma characterizes dominated tensegrity frameworks in terms of the matrix PP^T (which is often called the Gram matrix of the configuration $\mathbf{p}$).

Lemma 8.3.13. *Let $(G, \mathbf{p})$ and $(G, \mathbf{p}')$ be two tensegrity frameworks in $\mathbb{R}^r$ and $\mathbb{R}^s$ of the same labeled graph G. Then $(G, \mathbf{p}')$ is dominated by $(G, \mathbf{p})$ if and only if*

$$P'P'^T - PP^T = \mathcal{E}(y) + x\mathbf{1}^T + \mathbf{1}x^T, \tag{8.3.9}$$

for some $y = (y_{ij})$ and $x = (x_i) \in \mathbb{R}^n$ where $y_{ij} \geq 0$ for all $\{i, j\} \in C$ and $y_{ij} \leq 0$ for all $\{i, j\} \in S$.

Proof. Let $\mathcal{L} = \text{span}\,\{F^{ij} : \{i, j\} \in B\}$. Then it follows from Lemmas 8.3.6 and 8.3.7 that $\{E^{ij} : \{i, j\} \in \overline{E} \cup C \cup S\} \cup \{L^i : i = 1, \ldots, n\}$ is a basis for $\mathcal{L}^\perp$, the orthogonal complement of $\mathcal{L}$ in $\mathcal{S}_n$. Now, by equation (8.3.7), $(G, \mathbf{p}')$ is dominated by $(G, \mathbf{p})$ if and only if

$$\langle F^{ij}, P'P'^T - PP^T\rangle = 0 \quad \text{for all } \{i, j\} \in B, \tag{8.3.10}$$

$$\langle F^{ij}, P'P'^T - PP^T\rangle \leq 0 \quad \text{for all } \{i, j\} \in C, \tag{8.3.11}$$

$$\langle F^{ij}, P'P'^T - PP^T\rangle \geq 0 \quad \text{for all } \{i, j\} \in S. \tag{8.3.12}$$

But (8.3.10) holds if and only if $(P'P'^T - PP^T) \in \mathcal{L}$, i.e., $P'P'^T - PP^T = \sum_{\{i,j\}\in\overline{E}\cup C\cup S} y_{ij}E^{ij} + \sum_{i=1}^n x_i L^i = \mathcal{E}(y) + xe^T + ex^T$ for some y and x. On the other hand, (8.3.11) holds if and only if $y_{ij} \geq 0$ for all $\{i, j\} \in C$ since, for $\{k, l\} \in C$, it follows from Lemma 8.3.6 that

$$\langle F^{kl}, P'P'^T - PP^T\rangle = \sum_{\{i,j\}\in C} \langle F^{kl}, E^{ij}\rangle y_{ij} = -2y_{kl} \leq 0.$$

Similarly, (8.3.12) holds if and only if $y_{ij} \leq 0$ for all $\{i, j\} \in S$. ☐

The following result reveals a crucial property that a proper PSD stress matrix of a framework is a stress matrix for every dominated framework. Moreover, cables and struts with non-zero stresses behave as bars.

Theorem 8.3.14. *Let $(G, \mathbf{p})$ be a given tensegrity framework and Ω a proper PSD stress matrix of $(G, \mathbf{p})$. Suppose that $(G, \mathbf{p}')$ is a tensegrity framework dominated by $(G, \mathbf{p})$. Then the following holds.*

1. *Ω is a stress matrix for $(G, \mathbf{p}')$.*
2. *For every $\{i, j\} \in C \cup S$ such that $\omega_{ij} \neq 0$ we have $\|p'^i - p'^j\| = \|p^i - p^j\|$.*

Proof. Let $(G, \mathbf{p}')$ be a tensegrity framework dominated by $(G, \mathbf{p})$. Then, by equation (8.3.7) and definition,

$$\omega_{ij}\,\langle F^{ij}, P'P'^T - PP^T\rangle = 0 \qquad \text{for each } \{i, j\} \in B,$$

$$\omega_{ij}\,\langle F^{ij}, P'P'^T - PP^T\rangle \leq 0 \qquad \text{for each } \{i, j\} \in C,$$

$$\omega_{ij}\,\langle F^{ij}, P'P'^T - PP^T\rangle \leq 0 \qquad \text{for each } \{i, j\} \in S.$$

Therefore,

$$\langle \Omega \,, P'P'^T - PP^T \rangle = \sum_{\{i,j\} \in B \cup C \cup S} \omega_{ij} \, \langle F^{ij} \,, P'P'^T - PP^T \rangle \leq 0.$$

But $\Omega PP^T = 0$. Therefore, $\langle \Omega \,, P'P'^T \rangle \leq 0$. However, both $P'P'^T$ and Ω are positive semidefinite. Therefore, $\langle \Omega \,, P'P'^T \rangle = 0$ and hence $\Omega P'P'^T = \mathbf{0}$ by Lemma 2.5.3. It follows that Ω is a stress matrix of $(G, \mathbf{p}')$.

The above argument also implies that $\omega_{ij} \, \langle F^{ij} \,, P'P'^T - PP^T \rangle = 0$ for all $\{i,j\} \in E$. Hence, for $\{i,j\} \in C \cup S$ with $w_{ij} \neq 0$ we have $\langle F^{ij} \,, P'P'^T - PP^T \rangle = 0$, thus $\|p'^i - p'^j\| = \|p^i - p^j\|$. □

Using Theorem 8.3.14, the following theorem easily follows.

Theorem 8.3.15. *Let $(G, \mathbf{p})$ be a given tensegrity framework on n vertices in $\mathbb{R}^r$ with $r \leq n-2$ and let Ω be a proper PSD stress matrix of $(G, \mathbf{p})$ of rank $n-r-1$. Then $(G, \mathbf{p})$ is dimensionally rigid.*

Proof. Let $(G, \mathbf{p}')$ be a tensegrity framework of full dimension in $\mathbb{R}^s$ dominated by $(G, \mathbf{p})$. Then the dimension of the nullspace of Ω is $r+1$. Moreover, it follows from Theorem 8.3.14 that $\Omega P'P'^T = \mathbf{0}$. On the other hand, $\mathbf{1}^T \Omega = \mathbf{0}$, thus $\mathbf{1}$ is not in the column space of Ω. Also we have $\mathbf{1}^T P' = 0$ by Assumption 8.3.1. Therefore, rank $P' \leq r$ and hence, $(G, \mathbf{p})$ is dimensionally rigid. □

Proof of Theorem 8.3.1 Let $(G, \mathbf{p}')$ be a tensegrity framework in $\mathbb{R}^s$ dominated by $(G, \mathbf{p})$ and let P' be the configuration matrix of $(G, \mathbf{p}')$. Theorem 8.3.15 implies that we can assume that $s = r$. Indeed, since $(G, \mathbf{p})$ is dimensionally rigid, if $s > r$ we can transform $(G, \mathbf{p}')$ to a congruent framework in $\mathbb{R}^r$ and if $s < r$ we can consider $(G, \mathbf{p}')$ as embedded in $\mathbb{R}^r$. Also, by the proof of Theorem 8.3.15, we have that the columns of P' belong to the nullspace of $\begin{pmatrix} \Omega \\ \mathbf{1}^T \end{pmatrix}$. Then there exists an $r \times r$ matrix A such that $P' = PA$ since the columns of P form a basis of this null space. Hence, the configuration $\mathbf{p}'$ is obtained from $\mathbf{p}$ by an affine transformation. Hence, $(G, \mathbf{p}')$ is congruent to $(G, \mathbf{p})$ and thus $(G, \mathbf{p})$ is universally rigid. □

8.3.5 Affinely dominated tensegrity frameworks

By Theorem 8.3.1, the main task in proving Theorem 8.3.4 is to exclude affinely-dominated but non congruent frameworks. The next result which is implicit in [76] characterizes when this happens.

Lemma 8.3.16. *Let $(G, \mathbf{p})$ be a tensegrity framework in $\mathbb{R}^r$. Then there exists a tensegrity framework $(G, \mathbf{p}')$ affinely-dominated by, but not congruent to, $(G, \mathbf{p})$ if and only if there exists a non-zero symmetric $r \times r$ matrix Φ such that*

$$\langle F^{ij}, P\Phi P^T\rangle = 0 \quad \textit{for all } \{i,j\} \in B, \tag{8.3.13}$$
$$\langle F^{ij}, P\Phi P^T\rangle \leq 0 \quad \textit{for all } \{i,j\} \in C, \tag{8.3.14}$$
$$\langle F^{ij}, P\Phi P^T\rangle \geq 0 \quad \textit{for all } \{i,j\} \in S. \tag{8.3.15}$$

Proof. To prove the "only if" part, assume that $(G, \mathbf{p}')$ is affinely-dominated by, but not congruent to, $(G, \mathbf{p})$. Then $P' = PA$ for some $r \times r$ matrix A. Thus, by (8.3.7),

$$\|p'^i - p'^j\|^2 - \|p^i - p^j\|^2 = \langle F^{ij}, P(AA^T - I)P^T\rangle.$$

The result follows by setting $\Phi = AA^T - I$. Obviously, Φ is symmetric. Furthermore, A is not orthogonal since $(G, \mathbf{p}')$ is not congruent to $(G, \mathbf{p})$ thus $\Phi \neq \mathbf{0}$.

To prove the "if" part assume that there exists a non-zero symmetric matrix Φ satisfying (8.3.13)-(8.3.15). Then there exists a sufficiently small $\epsilon > 0$ such that $I + \epsilon\Phi$ is positive definite. Thus there exists an $r \times r$ nonsingular matrix A such that $AA^T = I + \Phi$. Hence,

$$\langle F^{ij}, P\Phi P^T\rangle = \langle F^{ij}, P(AA^T - I)P^T\rangle = \|p'^i - p'^j\|^2 - \|p^i - p^j\|^2.$$

Thus the result follows. □

If a proper stress matrix of $(G, \mathbf{p})$ is known, then Lemma 8.3.16 can be strengthened. (Also implicit in [76].)

Lemma 8.3.17. *Let $(G, \mathbf{p})$ be a tensegrity framework in $\mathbb{R}^r$ and let Ω be a proper stress matrix of $(G, \mathbf{p})$. Then there exists a tensegrity framework $(G, \mathbf{p}')$ affinely-dominated by, but non congruent to, $(G, \mathbf{p})$ if and only if there exists a nonzero symmetric $r \times r$ matrix Φ such that:*

$$\langle F^{ij}, P\Phi P^T\rangle = 0 \quad \textit{for all } \{i,j\} \in B \cup C^* \cup S^*, \tag{8.3.16}$$
$$\langle F^{ij}, P\Phi P^T\rangle \leq 0 \quad \textit{for all } \{i,j\} \in C^0, \tag{8.3.17}$$
$$\langle F^{ij}, P\Phi P^T\rangle \geq 0 \quad \textit{for all } \{i,j\} \in S^0, \tag{8.3.18}$$

where $C^* = \{\{i,j\} \in C : \omega_{ij} \neq 0\}$, $S^* = \{\{i,j\} \in S : \omega_{ij} \neq 0\}$, $C^0 = C \setminus C^*$ and $S^0 = S \setminus S^*$.

Proof. Let Φ be the matrix defined as in Lemma 8.3.16. By definition, $\Omega P \Phi P^T = \mathbf{0}$. Therefore,

$$\begin{aligned} \langle \Omega \,, P\Phi P^T \rangle &= \sum_{\{i,j\} \in E} \omega_{ij} \langle F^{ij} \,, P\Phi P^T \rangle \\ &= \sum_{\{i,j\} \in C \cup S} \omega_{ij} \langle F^{ij} \,, P\Phi P^T \rangle \\ &= 0, \end{aligned}$$

since $\langle F^{ij} \,, P\Phi P^T \rangle = 0$ for every $\{i,j\} \in B$. But $\omega_{ij} \langle F^{ij} \,, P\Phi P^T \rangle \leq 0$, for every $\{i,j\} \in C$, since $\omega_{ij} \geq 0$ and $\langle F^{ij} \,, P\Phi P^T \rangle \leq 0$. Similarly, $\omega_{ij} \langle F^{ij} \,, P\Phi P^T \rangle \leq 0$, for every $\{i,j\} \in S$. Therefore,

$$\omega_{ij} \langle F^{ij} \,, P\Phi P^T \rangle = 0 \quad \text{for each } \{i,j\} \in C \cup S.$$

Thus, $\langle F^{ij} \,, P\Phi P^T \rangle = 0 \quad$ for each $\{i,j\} \in C^* \cup S^*$ and the result follows. $\square$

The next result allows us to translate (8.3.16)-(8.3.18) into a condition on Gale matrices which is easier to deal with since a Gale matrix with nice property can be obtained from a stress matrix of maximal rank.

Lemma 8.3.18. *Let $(G, \mathbf{p})$, $\Omega, C^0, S^0, C^*, S^*$ be defined as in Lemma 8.3.17 and Z be any Gale matrix of $\mathbf{p}$. If there exists a tensegrity framework $(G, \mathbf{p}')$ affinely-dominated by, but non congruent to, $(G, \mathbf{p})$ then there exist a non-zero $y = (y_{ij}) \in \mathbb{R}^{|\overline{E}|+|C^0|+|S^0|}$ and $\xi = (\xi_i) \in \mathbb{R}^{\bar{r}}$ such that*

$$\mathcal{E}^*(y) Z = \mathbf{1} \xi^T, \tag{8.3.19}$$

where $\mathcal{E}^*(y) = \sum_{\{i,j\} \in \overline{E} \cup C^0 \cup S^0} y_{ij} E^{ij}$.

Proof. Assume that there exists a non-zero symmetric Φ satisfying (8.3.16)-(8.3.18). Let $\mathcal{L} = \text{span}\,\{F^{ij} : \{i,j\} \in B \cup C^* \cup S^*\}$. Then it follows from Lemmas 8.3.6 and 8.3.7 that $\{E^{ij} : \{i,j\} \in \overline{E} \cup C^0 \cup S^0\} \cup \{L^i : i = 1, \ldots, n\}$ is a basis for $\mathcal{L}^\perp$, the orthogonal complement of $\mathcal{L}$ in $\mathcal{S}_n$. Since $\langle F^{ij} \,, P\Phi P^T \rangle = 0$ for all $\{i,j\} \in B \cup C^* \cup S^*$ if and only if $P\Phi P^T \in \mathcal{L}^\perp$, we have,

$$P\Phi P^T = \sum_{\{i,j\} \in \overline{E} \cup C^0 \cup S^0} y_{ij} E^{ij} + \sum_{i=1}^{n} x_i L^i = \mathcal{E}^*(y) + x \mathbf{1}^T + \mathbf{1} x^T$$

for some y and x. Therefore, y and $\xi = -Z^T x$ is a solution of (8.3.19) since $P\Phi P^T Z = \mathbf{0} = \mathcal{E}^*(y) Z + \mathbf{1} x^T Z$. $\square$

8.3.6 Proof of main results

We need some more lemmas. The first one simplifies equation (8.3.19) when Z has a nice property.

Lemma 8.3.19. *Let Z be a $n \times \bar{r}$ matrix whose rows are indexed by $\{1, \dots, n\}$ and whose columns are indexed by $\{j_1, \dots, j_{\bar{r}}\}$ with the property*

$$z_{ij_k} = 0 \qquad \textit{for } \{i, j_k\} \in \overline{E} \cup C^0 \cup S^0.$$

Then the equation $\mathcal{E}^*(y)Z = \mathbf{1}\xi^T$ is equivalent to $\mathcal{E}^*(y)Z = \mathbf{0}$.

Proof. For every $k = 1, \dots, \bar{r}$,

$$\begin{aligned} \xi_{j_k} &= \sum_{i=1}^{n} \mathcal{E}^*(y)_{j_k i} z_{ij_k} \\ &= \sum_{i:\{i,j_k\}\in E} \mathcal{E}^*(y)_{j_k i} z_{ij_k} + \sum_{i:\{i,j_k\}\in \overline{E}} \mathcal{E}^*(y)_{j_k i} z_{ij_k} + \mathcal{E}^*(y)_{j_k j_k} z_{j_k j_k} \\ &= 0 \end{aligned}$$

since if $\{i, j_k\} \in B \cup C^* \cup S^*$ then $\mathcal{E}^*(y)_{ij_k} = 0$ while if $\{i, j_k\} \in \overline{E} \cup C^0 \cup S^0$ then $z_{ij_k} = 0$. Therefore $\mathcal{E}^*(y)Z = 0$ holds. The converse is trivial. $\square$

When the stress matrix Ω has maximal rank then a Gale matrix with the nice property as in Lemma 8.3.19 can be obtained from Ω.

Lemma 8.3.20 (Alfakih and Ye [8])**.** *Let $(G, \mathbf{p})$ be a tensegrity framework in $\mathbb{R}^r$ such that there exists a stress matrix Ω of $(G, \mathbf{p})$ with* $\operatorname{rank} \Omega = n - r - 1$*. Then every $\bar{r} = n - r - 1$ linearly independent columns indexed by $j_1, \dots, j_{\bar{r}}$ of Ω is a Gale matrix $\hat{Z}$ of $\mathbf{p}$. Moreover, this Gale matrix $\hat{Z}$ has the property that $\hat{z}_{ij_k} = 0$ if $\{i, j_k\} \in \overline{E} \cup C^0 \cup S^0$. Here the rows of $\hat{Z}$ are indexed by $\{1, \dots, n\}$ while the columns of $\hat{Z}$ are indexed by $j_1, \dots, j_{\bar{r}}$.*

Proof. A matrix $\hat{Z}$ obtained as in the lemma is obviously full column rank and satisfies $\hat{z}_{ij_k} = 0$ if $\{i, j_k\} \in \overline{E} \cup C^0 \cup S^0$. Moreover, since $P^T\Omega = \mathbf{0}$ we have $P^T\hat{Z} = \mathbf{0}$. Hence $\hat{Z}$ is a Gale matrix of $\mathbf{p}$. $\square$

We will prove the following result which is in fact stronger than Theorem 8.3.4.

Theorem 8.3.21. *Let $(G, \mathbf{p})$ be a tensegrity framework on n vertices in $\mathbb{R}^r$ with $r \le n - 2$ that satisfies the following conditions.*

1. *$(G, \mathbf{p})$ has a PSD proper stress matrix Ω with* $\operatorname{rank} \Omega = n - r - 1$.

2. *For each $i \in \{1, \ldots, n\}$, the set $\{p^j : j \in \{i\} \cup N^*(i)\}$ affinely spans $\mathbb{R}^r$, where $N^*(i) = \{j \in V : \{i, j\} \in B \cup C^* \cup S^*\}$*

Then $(G, \mathbf{p})$ is universally rigid.

Proof. Suppose that $(G, \mathbf{p})$ is a framework satisfying the conditions in the theorem. By Theorem 8.3.1 it is sufficient to show that there does not exist a tensegrity framework $(G, \mathbf{p}')$ in $\mathbb{R}^r$ affinely-dominated by but non congruent to $(G, \mathbf{p})$. By Lemma 8.3.18 it is equivalent to showing that, for a Gale matrix Z, the equation $\mathcal{E}^*(y)Z = \mathbf{1}\xi^T$ has the unique solution $y = \mathbf{0}$.

Let $\hat{Z}$ be the Gale matrix in Lemma 8.3.20. For this matrix $\hat{Z}$, the equation $\mathcal{E}^*(y)\hat{Z} = e\xi^T$ is equivalent to $\mathcal{E}^*(y)\hat{Z} = \mathbf{0}$ by Lemma 8.3.19. Therefore, in the following we show that $\mathcal{E}^*(y)\hat{Z} = \mathbf{0}$ implies $y = \mathbf{0}$.

For each vertex $i \in \{1, \ldots, n\}$, let $J_i = \{i\} \cup N^*(i)$. The set $\{p^j : j \in J_i\}$ affinely spans $\mathbb{R}^r$, for every $i \in \{1, \ldots, n\}$, from the second assumption. Then by Corollary 8.3.12, the rows of $\hat{Z}_{\bar{J}_i}$ are linearly independent.

Let $(\hat{z}^j)^T$ be the jth row of $\hat{Z}$. Since $\mathcal{E}^*(y)\hat{Z} = \mathbf{0}$, for every $i \in \{1, \ldots, n\}$ we have $\sum_{j=1}^n \mathcal{E}^*(y)_{ij}(\hat{z}^j)^T = \mathbf{0}$, which is equivalent to $\sum_{j \in \bar{J}_i} \mathcal{E}^*(y)_{ij}(\hat{z}^j)^T = \mathbf{0}$, as $\mathcal{E}^*(y)_{ij} = 0$ for $j \in J_i$. The linear independence of the rows of $\hat{Z}_{\bar{J}_i}$ implies that $\mathcal{E}^*(y)_{ij} = 0$ for $j \in \bar{J}_i$, for every $i \in \{1, \ldots, n\}$. Therefore $y = \mathbf{0}$. This completes the proof. □

The following result indicates that, under the assumption about the stress matrix in Theorem 8.3.21, the general position condition implies the spanning condition.

Lemma 8.3.22. *Let $(H, \mathbf{p})$ be a tensegrity framework on n vertices in $\mathbb{R}^r$ which is of full dimension. Suppose that H has a stress matrix Ω of rank $n - r - 1$. Suppose further that, for every vertex $i \in \{1, \ldots, n\}$, the points p^j for all $j \in \{i\} \cup N_H(i)$ are in general position, where $N_H(i)$ denotes the set of neighbors of i in H. Then, for each $i \in \{1, \ldots, n\}$, the set $\{p^j : j \in \{i\} \cup N_H(i)\}$ affinely spans $\mathbb{R}^r$.*

Proof. It will suffice to show that for each $i \in \{1, \ldots, n\}$ we have $|N(i)| \geq r + 1$. Suppose by contradiction that there exists a vertex $i \in \{1, \ldots, n\}$ with at most r neighbors. Then since $\{p^j : j \in \{i\} \cup N_H(i)\}$ is in general position in $\mathbb{R}^r$, the vectors $p^j - p^i$ for $j \in N_H(i)$ are linearly independent. On the other hand, since Ω is a stress matrix for $(H, \mathbf{p})$, $\sum_{j \in N_H(i)} \omega_{ij}(p^i - p^j) = 0$ holds. Therefore, $\omega_{ij} = 0$ for every $j \in N_H(i)$. Let $H' = H - i$ be the graph obtained from H by deleting the vertex i and all its incident edges. Let $\mathbf{p}'$ be obtained by restricting $\mathbf{p}$ on

$\{1, \ldots, n\} \setminus \{i\}$. By deleting the ith row and ith column (which are all zeros) of Ω we obtain a stress matrix Ω' for the tensegrity framework $(H', \mathbf{p}')$. This matrix Ω' obviously has the same rank as Ω and hence $\operatorname{rank} \Omega' = n - r - 1 > n' - r - 1$. This implies that the affine dimension of the tensegrity framework $(H', \mathbf{p}')$ is less than r by Lemma 8.3.9. Moreover, this affine dimension can not be less than $r - 1$ since $(G, \mathbf{p})$ is of full dimension in $\mathbb{R}^r$. Therefore,

$$(H', \mathbf{p}') \text{ is a tensegrity framework of affine dimension } r - 1. \tag{8.3.20}$$

Since $\operatorname{rank} \Omega' = n - r - 1 > 0$, there must be some non-zero stress. Suppose that k is a vertex incident to an edge with a non-zero stress. Then $d_{H'}(k) \geq r$, otherwise, the same argument as in the beginning of the proof implies that all stresses on the edges incident to k in H' are zero, a contradiction. Let $N_{H'}(k)$ denote the set of neighbors of k in H'. Then, by the assumption of the lemma,$\{p^j : j \in k \cup N_{H'}(k)\}$ is a set of at least $r + 1$ points in general position in $\mathbb{R}^r$, a contradition to (8.3.20). The lemma follows. $\square$

Now we are ready to prove the main theorems.

Proof of Theorem 8.3.4

Let $(G, \mathbf{p})$ be a tensegrity framework in $\mathbb{R}^r$ satisfying the conditions in Theorem 8.3.4. Let $H = (V, B \cup C^* \cup S^*)$. Then it is obvious that Ω is also a PSD stress matrix for $(H, \mathbf{p})$. By Lemma 8.3.22, in $(H, \mathbf{p})$, for each $v \in V$, $\{p^j : j \in \{i\} \cup N_H(i)\}$ affinely spans $\mathbb{R}^r$. Therefore the tensegrity framework $(H, \mathbf{p})$ satisfies all the conditions in Theorem 8.3.21 and so $(H, \mathbf{p})$ is universally rigid. Thus $(G, \mathbf{p})$ is universally rigid. $\square$

Proof of Theorem 8.3.5

Let $(G, \mathbf{p})$ be a bar framework satisfying the conditions in Theorem 8.3.5. Then $(G, \mathbf{p})$, regarded as a tensegrity framework without cables and struts, obviously satifies the conditions in Theorem 8.3.21. Therefore, $(G, \mathbf{p})$ is universally rigid. $\square$

8.3.7 Counter-example and conjectures

Having Theorem 8.3.4 in hand, a natural question is whether the converse holds, that is, does every universally rigid framework in general position in $\mathbb{R}^r$ has a

PSD stress matrix of maximal rank. When "general" is replaced by "generic", the answer is affirmative for bar frameworks (Gortler and Thurston [45], cf. Theorem 3.2.7).

It is known that the answer is also affirmative for bar frameworks in general position when the underlying graphs belong to some special classes. A chordal graph is a graph without induced cycles of length greater than 3. A k-lateration graph is a graph that can be constructed from a complete graph on k vertices by recursively adding a new vertex and k edges incident to this vertex.

Theorem 8.3.23 (Alfakih [5], Alfakih, Taheri and Ye [7])**.** *Let G be a chordal graph or an $(r+1)$-lateration graph on $n \geq r+2$ vertices. If $(G, \mathbf{p})$ is a framework in general position in $\mathbb{R}^r$ which is universally rigid then $(G, \mathbf{p})$ has a PSD stress matrix of rank $n-r-1$.*

These results, unfortunately, can not be extended to all bar frameworks in general position. In the following we consider an example of a universally rigid bar framework in general position in $\mathbb{R}^2$ without any PSD stress matrices of maximal rank.

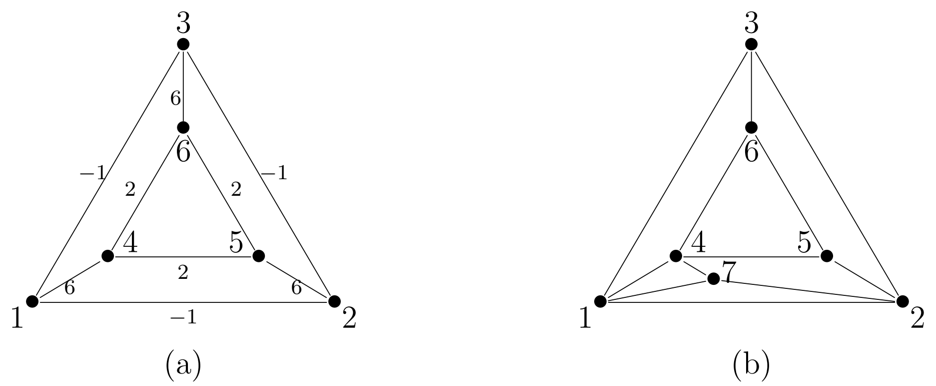


Figure 8.10: (a) A framework in $\mathbb{R}^2$ with a unique equilibrium stress.
(b) A new vertex 7 is added.

Consider a framework $(G, \mathbf{p})$ as in Figure 8.10(a) where the vertices $1, 2, 3$ form a regular triangle, the vertices $4, 5, 6$ also form a regular triangle with edges parallel to edges of $1, 2, 3$ and of half edge length. Furthermore 4 (resp., 5 and 6) lies on the median from 1 (resp., 2 and 3). Note that then the distance from 4 to 1 is one-third of the median length. It is not difficult to see that this framework has a unique non-zero equilibrium stress (up to a scalar multiple). Scaling this equilibrium stress so that the stress on the edge $\{1, 2\}$ is -1, we can easily deduce the stress on the other edges as indicated in the figure. The corresponding stress matrix is

$$\Omega = \begin{pmatrix} 4 & 1 & 1 & -6 & 0 & 0 \\ 1 & 4 & 1 & 0 & -6 & 0 \\ 1 & 1 & 4 & 0 & 0 & -6 \\ -6 & 0 & 0 & 10 & -2 & -2 \\ 0 & -6 & 0 & -2 & 10 & -2 \\ 0 & 0 & -6 & -2 & -2 & 10 \end{pmatrix}.$$

It is also easy to check that $\operatorname{rank}\Omega = 3$. The positive semidefiniteness can also be easily checked by calculating the first 3 leading principal minors (Lemma 2.5.2). The framework $(G, \mathbf{p})$ is obviously in general position, therefore, $(G, \mathbf{p})$ is a universally rigid framework in $\mathbb{R}^2$ by Theorem 8.3.3.

Now extend $(G, \mathbf{p})$ to a framework $(G', \mathbf{p}')$ in $\mathbb{R}^2$ by adding a new vertex 7 incident to $1, 2$ and 4 such that $(G', \mathbf{p}')$ is in general position as in Figure 8.10b. Then since $(G, \mathbf{p})$ is universally rigid, $(G', \mathbf{p}')$ must also be universally rigid. Suppose that (ω'_{ij}) is a non-zero stress matrix for $(G', \mathbf{p}')$. Then one can see that $\omega'_{1,3}$ must be non-zero. So we can suppose that $\omega'_{1,3} = 1 = \omega_{1,3}$. However, then $\omega'_{3,2} = \omega_{3,2}$ and $\omega'_{3,6} = \omega_{3,6}$ holds. Hence, $\omega'_{5,6} = \omega_{5,6}$ and $\omega'_{4,6} = \omega_{4,6}$. Thus again, $\omega'_{5,2} = \omega_{5,2}$ and $\omega'_{4,5} = \omega_{4,5}$. In $(G, \mathbf{p})$, the equilibrium at the vertex 4 yields $\omega_{4,1}(p^4 - p^1) + \omega_{4,5}(p^4 - p^5) + \omega_{4,6}(p^4 - p^6) = 0$. Therefore in $(G', \mathbf{p}')$, the vector $\omega'_{4,5}(p'^4 - p'^5) + \omega'_{4,6}(p'^4 - p'^6) = \omega_{4,5}(p^4 - p^5) + \omega_{4,6}(p^4 - p^6)$ lies on the same line as $p'^4 - p'^1 = p^4 - p^1$. Hence the equilibrium at 4 in $(G', \mathbf{p}')$ implies that $\omega'_{4,7} = 0$ as 7 does not lie on the line $1, 4$. We also deduce that $\omega'_{1,7} = \omega_{2,7} = 0$. So the only non-zero equilibrium stress of $(G', \mathbf{p}')$ has stress 0 on every edge incident to the vertex 7. Therefore, the corresponding stress matrix Ω' is obtained from Ω by adding an all-zero column and an all-zero row. The rank of this matrix is obviously that of Ω which is $3 < 7 - 2 - 1$.

We believe that the existence of a PSD matrix of maximal rank is necessary for bar frameworks in general position on the line, since a universally rigid framework on the line must be infinitesimally rigid.

Conjecture 8.3.24. *Let $(G, \mathbf{p})$ be a bar framework on $n \geq 3$ vertices in general position in $\mathbb{R}^1$. If $(G, \mathbf{p})$ is universally rigid then it has a PSD stress matrix of rank $n - 2$.*

We also conjecture that Theorem 3.2.7 can be extended to generic tensegrity frameworks.

Conjecture 8.3.25. *Let $(G, \mathbf{p})$ be a generic tensegrity framework on n vertices in $\mathbb{R}^r$ for some $r \leq n-2$. If $(G, \mathbf{p})$ is universally rigid then $(G, \mathbf{p})$ has a proper stress matrix of rank $n-r-1$.*

Conclusion

Although coming into the scene with different motivations and being often dealed with different techniques, infinitesimal rigidity, global rigidity and universal rigidity are tightly related. In this thesis we have treated a wide range of problems concerning these different kinds of rigidity in various types of frameworks.

Chapter 4 presented our study of local rigidity in bar-joint frameworks from a matroidal viewpoint. We characterized abstract rigidity matroids as well as intersecting submodular functions inducing these matroids.

In Chapter 5, we developed combinatorial results in inductive constructions and decompositions of graphs. We provided inductive constructions of $(m, \mathbf{d})$-graded sparse graphs where different types of edges are subject to different sparsity conditions, and of $(\mathbf{b}, l)$-sparse graphs where the sparsity of subgraphs depends on their vertex sets. We also obtained a decomposition of $(m, \mathbf{d})$-graded sparse graphs into graded pseudoforests. These results generalized classic results of Nash-Williams on the union of edge-disjoint spanning trees. Furthermore, we considered the directed counterpart (of the dual) of the decomposition problem and generalized a result of Edmonds to packing of matroid-based rooted arborescences.

In Chapter 6, we used the results developed in Chapter 5 to obtain characterizations of the infinitesimal rigidity of several types of frameworks with mixed constraints: body-bar frameworks with bar-boundary, body-length-direction frameworks. We also investigated the effect of extension operations on direction-length graphs.

Chapter 7 studied the effect of 1-extensions on the global rigidity of direction-length frameworks in general dimension, extending an earlier result of Jordán and Jackson.

Chapter 8 presented our results on universal rigidity. First we obtained a complete characterization of the universal rigidity of complete bipartite frameworks on the line and proved that every bipartite graph (except the single edge) is not generically universally rigid in any dimension. Second, we provided sufficient conditions for the universal rigidity of tensegrity frameworks (and hence bar-joint frameworks) which may be in non-general position.

One of the most challenging open problem which is of great importance in rigidity theory is certainly that on the combinatorial characterization of 3-dimensional generic local and global rigidity. Though some partial results have been obtained for special classes of graphs such as molecular graphs, this problem remains difficult to tackle for the lack of good conjectures.

A less studied area so far is universal rigidity. This area is particularly interesting firstly because known results are few even in dimension one. Secondly many open questions can be formed. Furthermore, to characterize universal rigidity combinatorially, even on the line, one would need a deep understanding of the motions of frameworks between different dimensions.

Bien qu'introduites pour des motifs différents et traitées à l'aide de techniques différentes, les rigidités infinitésimale, globale et universelle sont étroitement liées. Dans cette thèse nous avons traité une large étendue de problèmes concernant ces différentes sortes de rigidité, dans des types de charpentes variées.

Le Chapitre 4 a présenté notre étude de la rigidité locale des charpentes barres-et-joints à partir d'un point de vue matroïdal. Nous avons caractérisé les matroïdes de la rigidité abstraite ainsi que les fonctions sous-modulaires intersectantes induisant ces matroïdes.

Dans le Chapitre 5, nous avons développé des résultats combinatoires sur des constructions inductives et des décompositions de graphes. Nous avons fourni des constructions inductives de graphes $(m, \mathbf{d})$-graded sparses où différents types d'arêtes sont sujets à des conditions différentes de densité, et de graphes $(\mathbf{b}, l)$-sparse où la densité des sous-graphes dépend de l'ensemble de leur sommets. Nous avons aussi obtenu une décomposition de graphes $(m, \mathbf{d})$-graded sparse en des pseudo-forêts. Ces résultats ont généralisé des résultats classiques de Nash-Williams sur les unions d'arbres couvrants à arrêtes disjointes. De plus, nous avons considéré la contrepartie orientée (du dual) du problème de décomposition et avons généralisé un résultat d'Edmonds pour le packing de matroid-based rooted arborescences.

Au Chapitre 6, nous avons utilisé les résultats développés au Chapitre 5 pour obtenir la caractérisation de la rigidité infinitésimale de plusieurs types de charpentes avec des contraintes mixtes: charpentes body-bar avec des frontières par des barres, charpentes de body-length-direction. Nous avons aussi étudié les effets d'opérations d'extension sur des graphes direction-length.

Au Chapitre 7 nous avons étudié les effets de 1-extensions sur la rigidité globale des charpentes de direction-length en dimension quelconque, étendant un résultat antérieur de Jordán et Jackson.

Le Chapitre 8 a présenté nos résultats sur la rigidité universelle. Tout d'abord, nous avons obtenu une caractérisation complète de la rigidité universelle de charpentes biparties complètes sur la ligne et avons prouvé que tout graphe biparti (excepté celui réduit à une seule arête) n'est pas génériquement universellement rigide en toute dimension. En second lieu, nous avons fourni des conditions suffisantes pour la rigidité universelle de charpentes de tensegrité (et par conséquent de charpentes barres-et-joints) qui peuvent être en position non-générale.

Un des problèmes ouverts le plus ardu qui est d'une grande importance dans la théorie de la rigidité est certainement la caractérisation combinatoire de la rigidité générique locale et globale en dimension 3. Bien que des résultats partiels ont été obtenus pour des classes spéciales de graphes tels que les graphes moléculaires, ce problème reste difficile à résoudre du fait du manque de bonnes conjectures.

Un domaine moins étudié jusqu'à présent est celui de la rigidité universelle. Ce domaine est particulièrement intéressant d'une part car il y a très peu de résultats connus en dimension 1. Ensuite, de nombreuses questions ouvertes peuvent être formulées. Finalement, afin de caractériser combinatoirement la rigidité universelle, même sur la ligne, la compréhension profonde des déplacements des charpentes entre des dimensions différentes est nécessaire.

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Abstract

The theory of rigidity studies the uniqueness of realizations of graphs, i.e., frameworks. Originally motivated by structural engineering, rigidity theory nowadays finds applications in many important problems such as predicting protein flexibility, Computer-Aided Design, sensor network localization, etc. The present thesis treats a wide range of problems concerning different kinds of rigidity, corresponding to different scopes of uniqueness (local/infinitesimal, global and universal), in various types of frameworks. First, we develop results in inductive construction and decomposition of graphs with mixed sparsity conditions as well as results on the packing of arborescences with matroidal constraints. These results are then used to obtain characterizations of infinitesimal rigidity in frameworks with mixed constraints. We also investigate the effect of extension operations on frameworks and extend a known result on the global rigidity preservation of 1-extension on direction-length frameworks in dimension two to all dimensions. For universal rigidity, where little is known, we obtain a complete characterization for the class of complete bipartite frameworks on the line. We also generalize a sufficient condition for the universal rigidity of frameworks by allowing non-general positions.

Résumé

La théorie de la rigidité étudie l'unicité des réalisations des graphes, i.e., des charpentes. Initialement motivée par l'ingénierie des structures, la théorie de la rigidité trouve aujourd'hui des applications dans plusieurs domaines importants comme la prédiction de la flexibilité des protéines, la conception assistée par ordinateur, la localisation dans les réseaux des capteurs, etc. Cette thèse traite une grande variété de problèmes concernant différents types de rigidité, qui correspondent à différents niveaux d'unicité (locale/infinitésimale, globale et universelle) dans des modèles variés de charpentes. D'abord, nous développons des résultats sur la construction récursive et la décomposition des graphes avec des conditions mixtes de sparsité ainsi que des résultats sur le packing des arborescences avec des contraintes de matroïde. Ces résultats sont alors utilisés pour obtenir des caractérisations de la rigidité infinitésimale des charpentes avec des contraintes mixtes. Nous étudions aussi l'effet des opérations d'extension sur des charpentes et étendons un résultat connu sur la préservation de la rigidité globale d'1-extension dans les charpentes à direction et à longueur de la dimension deux aux dimensions supérieures. Pour la rigidité universelle, un sujet que l'on connait très peu, nous obtenons une caractérisation complète pour la classe des charpentes biparties complètes sur la ligne. Nous généralisons aussi une condition suffisante pour la rigidité universelle des charpentes en permettant des positions non générales.